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CS300

Fall 2018, Assignment #3

PROBLEM 2 (2+2+2+1+1+1+2P)**:**
Convolution is the following operation on vectors of same dimension n:

$$(\vec{a},\vec{b}) \;=\; \big((a_0,\ldots,a_{n-1}),(b_0,\ldots,b_{n-1})\big) \;\mapsto\; \Big(\sum_{j=0}^{n-1} a_j \cdot b_{k-j \bmod n} : k=0,\ldots,n-1\Big)$$

a) Design and analyze an algorithm for convolution using a *sub*quadratic number $T(n)$ of arithmetic operations (no transcendental constants). What asymptotic growth do you achieve? Hint: Consider the product of polynomials like $A(X) := a_0 + a_1X + a_2X^2 + \ldots$ and $B(X)$.

b) Verify the following equation for invertible 2×2 *block* matrices, i.e. with *non*-commuting (but invertible) entries: $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} =$

$$= \begin{pmatrix} A^{-1} + A^{-1}\cdot B\cdot(D-C\cdot A^{-1}\cdot B)^{-1}\cdot C\cdot A^{-1} & -A^{-1}\cdot B\cdot(D-C\cdot A^{-1}\cdot B)^{-1} \\ -(D-C\cdot A^{-1}\cdot B)^{-1}\cdot C\cdot A^{-1} & (D-C\cdot A^{-1}\cdot B)^{-1} \end{pmatrix}$$

c) Devise and analyze a divide-and-conquer algorithm for matrix inversion using (b). Describe the temporary calculations and recursive calls it makes, to justify your recurrence for the number $T(n)$ of arithmetic operations employed.

d) To justify the lecture's analysis of binary search prove that, for any integers $a < b$ and for $n := \lfloor (a+b)/2 \rfloor$, it holds $b-(n+1) \leq \lfloor (b-a)/2 \rfloor$ and $n-a \leq \lfloor (b-a)/2 \rfloor$.

e) Let $(X,\leq)$ be linearly ordered, $x_1,\ldots,x_N \in X$ and $K \in \{1,\ldots,N\}$. Prove that there exists some $M \in \{1,\ldots,N\}$, called a *K-quantile*, such that

$$\#\{n : x_n < x_M\} \;<\; K \;\leq\; \#\{n : x_n \leq x_M\} \tag{1}$$

f) Continuing (e), suppose $x_1 \leq \cdots \leq x_N$. Prove that K constitutes a K-quantile.

g) Adapt *BubbleSort* such that it does not modify the input array $x[1,\ldots N]$ but instead returns a permutation $\pi[1,\ldots N]$ with $x\big[\pi[1]\big] \leq \cdots \leq x\big[\pi[N]\big]$.