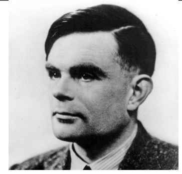
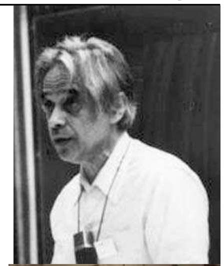
Computing Real Numbers





Theorem: For $r\in\mathbb{R}$, the following are equivalent:

Call $r\in\mathbb{R}$ computable if

a) r has a decidable binary expansion

b) There exists an algorithm computing a sequence $(a_n)\subseteq\mathbb{Z}$ with $|r-a_n/2^n|\leq 2^{-n}$.

c) There exist three algorithms computing sequences $(a_n),(b_n),(c_n)\subseteq\mathbb{Z}$ with $|r-a_n/b_n|\leq 1/c_n\to 0$

Ernst Specker (1949): $(c)^H \Leftrightarrow (d)$

d) There is an algorithm computing $(q_n)\subseteq\mathbb{Q}$ s.t. $q_n\to r$.

$H=\{\ \langle\mathcal{A},\underline{x}\rangle$: algorithm $\mathcal{A}$ terminates on input $\underline{x}\ \}\subseteq\mathbb{N}$

Computing Real Sequences

Call $(r_j)\subseteq\mathbb{R}$ computable iff an algorithm can print, on input $\langle n,j\rangle\in\mathbb{N}$, some $a\in\mathbb{Z}$ with $|r_j-a/2^n|\leq 2^{-n}$.

Proposition: If (r_j) is a computable sequence s.t. $|r_j-r_i|\leq 2^{-j}+2^{-i}$, then $\lim_j r_j$ is a computable real.

Example of a computable sequence $(r_j)\subseteq[0,1]$ such that $\{\ j : r_j\neq 0\ \} = H$, the Halting problem.

In numerics, don't test for (in-)equality!

$x\in\mathbb{R}$ computable $\Leftrightarrow$ $|x-a_n/2^n|\leq 2^{-n}$ for recursive $(a_n)\subseteq\mathbb{Z}$

Call $f:\subseteq\mathbb{R}\to\mathbb{R}$ computable iff an algorithm can convert any $(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\leq 2^{-m}$ into some $(b_n)\subseteq\mathbb{Z}$ with $|f(x)-b_n/2^n|\leq 2^{-n}$

Computing Real Functions

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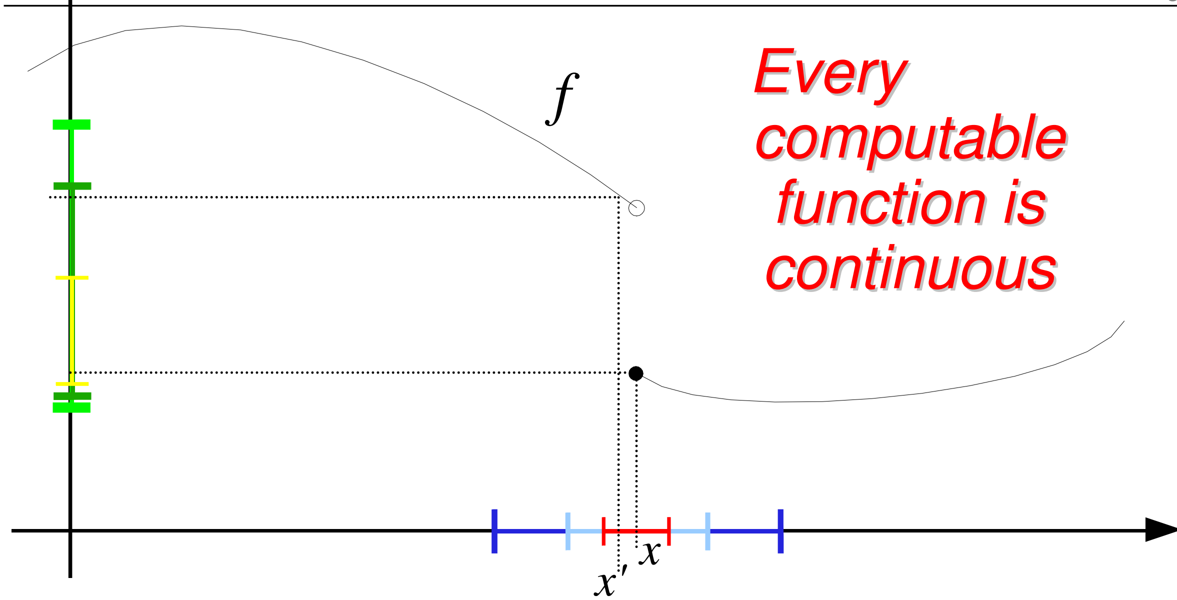
CS493 M. Ziegler



Every computable function is continuous

Call $f:\subseteq\mathbb{R}\to\mathbb{R}$ computable iff an algorithm can convert any $(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\leq 2^{-m}$ into some $(b_n)\subseteq\mathbb{Z}$ with $|f(x)-b_n/2^n|\leq 2^{-n}$

Mathematical Recap

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- continuous function on dense dom., separability
- continuous vs. uniformly continuous $f:\subseteq\mathbb{R}\to\mathbb{R}$
- three notions of compactness
- max. of continuous function on compact set
- Weierstrass Approximation Theorem
- co-r.e. $L\subseteq\mathbb{N}$

modulus of (unif) continuity

$|x-y|\leq 2^{-\mu(n)} \Rightarrow |f(x)-f(y)|\leq 2^{-n}$

König's Lemma: $X\subseteq\mathbb{Z}^{\mathbb{N}}$ is compact iff it is closed and the „tree" $X^* := \{\ \bar{a}\in\mathbb{Z}^* \mid \exists\underline{b}\in\mathbb{Z}^{\mathbb{N}} : \bar{a}\underline{b}\in X\ \}$ of finite initial segments is finitely branching.

Computable Weierstrass Theorem

Theorem: For $f:[0,1]\to\mathbb{R}$ the following are equivalent:

a) There is an algorithm converting any $\underline{a}=(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\le 2^{-m}$, into $(b_n)\in\mathbb{Z}$ with $|f(x)-b_n/2^n|\le 2^{-n}$

b) There is an algorithm printing a sequence (of deg.s and coefficient lists of) $(P_n)\subseteq\mathbb{D}[X]$ with $\|f-P_n\|_\infty\le 2^{-n}$

c) The real sequence $f(q)$, $q\in\mathbb{D}\cap[0,1]$, is computable $\wedge$ f admits a computable modulus of (unif) continuity

$|x-y|\le 2^{-\mu(n)} \Rightarrow |f(x)-f(y)|\le 2^{-n}$

Proof: b $\Leftrightarrow$ c $\Rightarrow$ a $\Rightarrow$ c

Call $(r_j)\subseteq\mathbb{R}$ computable iff an algorithm can print, on input $n,j\in\mathbb{N}$, some $q=a/2^n\in\mathbb{D}_n$ with $|r_j-q|\le 2^{-n}$.
$\mathbb{D} := \bigcup_n \mathbb{D}_n$, $\quad \mathbb{D}_n := \{\, a/2^n : a\in\mathbb{Z} \,\}$

Compactness in Real Computation

Lemma: Suppose $\mathcal{A}$ converts any $\underline{a}=(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\le 2^{-m}$, $x\in[0;1]$, to (b_n) s.t. $|f(x)-b_n/2^n|\le 2^{-n}$.

a) $t_{\mathcal{A}}(n):\underline{a}\to$#steps $\mathcal{A}$ makes on input $\underline{a}$ to print b_n is locally constant (=continuous) a function

b) giving rise to a *modulus of* local *continuity* of f:

$$\forall x\ \exists \underline{a}:\quad |x-x'|\le 2^{-t(n,\underline{a})-1} \quad\Rightarrow\quad |f(x)-f(x')|\le 2^{-n+1}$$

c) Its domain $\{\ \underline{a}\in\mathbb{Z}^{\mathbb{N}}: \exists x\in[0;1]\ \forall m: |x-a_m/2^m|\le 2^{-m}\ \}$ is compact in Baire Space $\mathbb{Z}^{\mathbb{N}}$ wrt $d(\underline{a},\underline{b})=2^{-\min\{n:a_n\ne b_n\}}$

d) and its set of finite initial segments is decidable
$\{\underline{\bar{a}}\in\mathbb{Z}^{\mathbb{N}}: m\in\mathbb{N},\ \forall i,j: -1\le a_j\le 1+2^j \wedge |a_i/2^i-a_j/2^j|\le 2^{-i}+2^{-j}\ \}$

e) $t_{\mathcal{A}}:\mathbb{N}\ni n\to\max_{\underline{a}} t_{\mathcal{A}}(n,\underline{a})$ is well-def and computable

Examples of Computable Real Functions

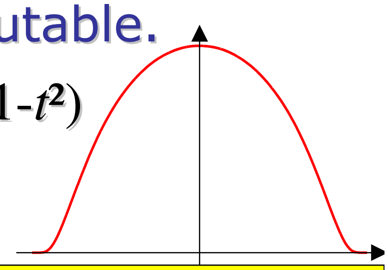
a) f computable $\Rightarrow$ so is any restriction of f

b) exp, sin, cos, ln are computable functions

c) Let $f \in C[0,1]$ be computable. Then so are
$\int f: x \to \int_0^x f(t)\,dt$ and $\max(f): x \to \max\{f(t): t \le x\}$.

d) For computable $f:[-1,0]\to\mathbb{R}$, $g:[0,1]\to\mathbb{R}$ with $f(0)=g(0)$, their join is computable.

e) C^∞ 'pulse' function $\varphi(t) = \exp(-t^2/1-t^2)$



To compute $f:\subseteq\mathbb{R}\to\mathbb{R}$: convert any sequence $(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m| \le 2^{-m}$ to $(b_n)\subseteq\mathbb{Z}$ with $|f(x)-b_n/2^n| \le 2^{-n}$

Myhill'71: uncomputable ∂ on $C^1[0,1]$

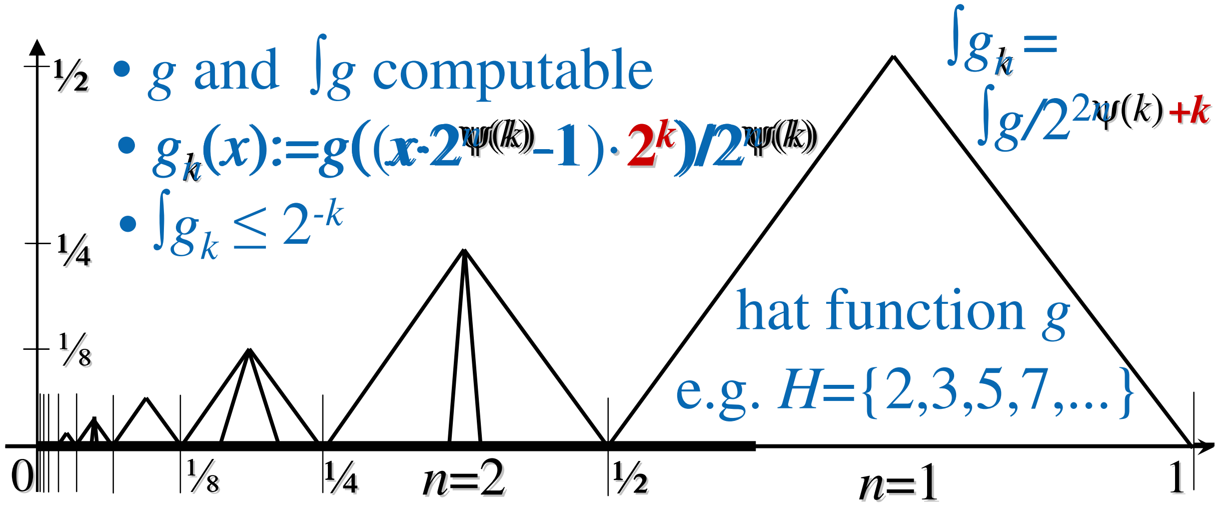
Recall computable bijection $\psi:\mathbb{N}\to H$

- g and $\int g$ computable
- $g_k(x) := g((x\cdot 2^{\psi(k)}-1)\cdot 2^k)/2^{\psi(k)}$
- $\int g_k \le 2^{-k}$

$\int g_k = \int g / 2^{2\psi(k)+k}$

hat function g

e.g. $H=\{2,3,5,7,\ldots\}$



$h' := \sum_{k\in\mathbb{N}} g_k$ continuous, incomputable,

yet $h := \int h' \in C^1[0;1]$ computable. q.e.d.