Recap Computing Real Numbers

**Theorem:** For \( r \in \mathbb{R} \), the following are equivalent:

a) \( r \) has a decidable binary expansion

b) There exists an algorithm computing a sequence \( (a_n) \subseteq \mathbb{Z} \) with \( |r - a_n/2^n| \leq 2^{-n} \).

c) There exist three algorithms computing sequences \( (a_n), (b_n), (c_n) \subseteq \mathbb{Z} \) with \( |r - a_n/b_n| \leq 1/c_n \rightarrow 0 \).

**Definition:** A WHILE+ program computes \( r \in \mathbb{R} \) in polytime iff, on input \( n \) after \( \leq \text{poly}(n) \) steps, it returns some \( a_n \in \mathbb{Z} \) with \( |r - a_n/2^n| \leq 2^{-n} \).

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Polytime-Computable Reals

**Example:** The following are polytime computable:

- sum, product, and reciprocal of polytime-computable reals
- every algebraic real
- some transcendental reals such as \( e=2.718.. \) or \( \pi \).

**Definition:** A WHILE+ program computes \( r \in \mathbb{R} \) in polytime iff, on input \( n \) after \( \leq \text{poly}(n) \) steps, it returns some \( a_n \in \mathbb{Z} \) with \( |r - a_n/2^n| \leq 2^{-n} \).
Computing Functions in Polytime

**Recap:** For $f: [0,1] \rightarrow \mathbb{R}$ the following are equivalent:

a) There is an algorithm converting any $a = (a_m) \subseteq \mathbb{Z}$ with $|x-a_m/2^m| \leq 2^{-m}$, into $(b_n) \subseteq \mathbb{Z}$ with $|f(x)-b_n/2^n| \leq 2^{-n}$

b) There is an algorithm printing a sequence (of deg.s and coefficient lists of) $(P_n) \subseteq \mathbb{D}[X]$ with $\|f-P_n\|_\infty \leq 2^{-n}$

c) The real sequence $f(q)$, $q \in \mathbb{D} \cap [0,1]$, is computable $\land f$ admits a computable modulus of continuity

**Theorem:** To approximate $|x-\frac{1}{2}|$ up error $2^{-n}$ requires polynomials of degree exponential in $n$.

**Definition:** A WHILE+ program computes $r \in \mathbb{R}$ in **polytime** iff, on input $n$ after $\leq \text{poly}(n)$ steps, it returns some $a_n \in \mathbb{Z}$ with $|r-a_n/2^n| \leq 2^{-n}$.

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Properties of Polytime Functions

**Def:** Computing $f: \subseteq \mathbb{R} \rightarrow \mathbb{R}$ in time $t(n)$ means to print, given $(a)_m \subseteq \mathbb{Z}$ with $|x-a_m/2^m| \leq 2^{-m}$ for $x \in \text{dom}(f)$, $(b_n) \subseteq \mathbb{Z}$ such that $|f(x)-b_n/2^n| \leq 2^{-n}$ within $t(n)$ steps.

Runtime may depend only on output precision $n$.

**Theorem:** If $f: [0;1] \rightarrow \mathbb{R}$ is computable, then so within bounded time $t(n)$ for some $t: \mathbb{N} \rightarrow \mathbb{N}$

**Theorem:** If $f: \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is computable in time $t(n)$, then $\mu(n):=t(n+1)+1$ is a modulus of continuity of $f$.

**Definition:** A WHILE+ program computes $r \in \mathbb{R}$ in **polytime** iff, on input $n$, after $\leq \text{poly}(n)$ steps, it returns some $a_n \in \mathbb{Z}$ with $|r-a_n/2^n| \leq 2^{-n}$.
Complexity of 1D Maximization

Fix polytime-comput. \( f: [0;1] \rightarrow [0;1] \) (⇒ continuous)

Max(\( f \)): \([0;1] \ni x \rightarrow \max \{ f(t): t \leq x \} \)

is computable in exponential time

is polytime-computable, provided that \( \mathcal{P} = \mathcal{NP} \)

if polytime for every (smooth) \( f \), then \( \mathcal{P} = \mathcal{NP} \):

**Thm** [Friedman&Ko’82]
To every \( L \in \mathcal{NP} \) exists
polytime \( C^\infty \) \( g_L: [0;1] \rightarrow \mathbb{R} \)
s.t. \([0;1] \ni x \rightarrow \max g_L|_{[0;x]} \)
again polytime iff \( L \in \mathcal{P} \)

\( \mu \) polyn. modulus of continuity

\( f \)

1/2^n

1/2^{\mu(n)}

\( g_L: t \rightarrow \sum_{\langle N,M \rangle \in V} \varphi(3tN^2-3M^2/MN)N^\ln N \)

\( \varphi(t) = \exp(-t^2/1-t^2) \)

\( C^\infty \) 'pulse' function

polytime computable

\( \mathcal{NP} \ni L = \{ N \in \mathbb{N} | \exists M < N : (N,M) \in V \} \) polytime

To every \( L \in \mathcal{NP} \) there exists a polytime computable \( C^\infty \) function \( g_L: [0;1] \rightarrow \mathbb{R} \) s.t.:
\([0;1] \ni x \rightarrow \max g_L|_{[0;x]} \) again polytime iff \( L \in \mathcal{P} \)
Complexity Conjectures in Numerics

Fix polytime \( f : [0;1] \rightarrow [0;1] \) \( \Rightarrow \) continuous

- \( \text{Max: } f \rightarrow \text{Max}(f) : x \rightarrow \max \{ f(t) : t \leq x \} \)
  - \( \text{Max}(f) \) computable in exponential time; \( \text{polyn.time-computable } \) iff \( \boldsymbol{P=NP} \)

- \( \int : f \rightarrow \int f : (x \rightarrow \int_0^x f(t) \, dt) \) \( \text{even for } f \in \mathcal{C}^\infty \)
  - \( \int f \) computable in exponential time; \( \text{polyn.time-computable } \) iff \( \boldsymbol{P=\#P} \)

- odesolve: \( C^1([0;1] \times [-1;1]) \exists f \rightarrow z : \dot{z}(t) = f(t,z), \ z(0)=0 \).
  - \( \text{PSPACE-"complete"} \)

- Solution to Poisson's Equation is classical and \( \#P-\text{"complete"} \)

\[ \Delta u = f \text{ on } B_2(0,1) \]
\[ u = 0 \text{ on } \partial B_2(0,1) \]

Perspective & Vision

- Computing on (σ-) compact metric spaces
- Rigorous Computability and Complexity Theory of partial differential equations (PDEs)
  - Why error bound \( 2^{-n} \) rather than \( 1/n \) ?
  - Why absolute errors rather than relative ?
  - Why inputs only by approximation ?

(Pure) Logic, Real/Complex/ Functional Analysis

Theoret. Computer Science

(Applied) Numerics/ Engineering