Recap Computing Real Numbers

Theorem: For $r \in \mathbb{R}$, the following are equivalent:

a) r has a decidable binary expansion

b) There exists an algorithm computing a sequence $(a_n) \subseteq \mathbb{Z}$ with $|r - a_n/2^n| \leq 2^{-n}$.

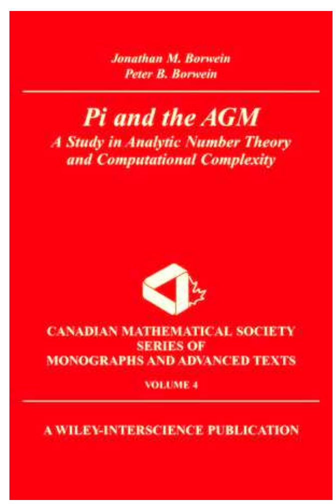
c) There exist three algorithms computing sequences $(a_n),(b_n),(c_n) \subseteq \mathbb{Z}$ with $|r - a_n/b_n| \leq 1/c_n \to 0$

Definition: A WHILE+ program **computes** $r \in \mathbb{R}$ **in polytime** iff, on input n **after $\leq$poly(n) steps,** it returns some $a_n \in \mathbb{Z}$ with $|r - a_n/2^n| \leq 2^{-n}$.

Polytime-Computable Reals

Example: The following are polytime computable:

- sum, product, and reciproce of polytime-computable reals
- every algebraic real
- some transcendental reals such as e=2.718.. or π.



Definition: A WHILE+ program **computes** $r \in \mathbb{R}$ **in polytime** iff, on input n **after $\leq$poly(n) steps,** it returns some $a_n \in \mathbb{Z}$ with $|r - a_n/2^n| \leq 2^{-n}$.

Computing Functions in Polytime

Recap: For $f:[0,1]\to\mathbb{R}$ the following are equivalent:

a) There is an algorithm converting any $\underline{a}=(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\leq 2^{-m}$, into $(b_n)\in\mathbb{Z}$ with $|f(x)-b_n/2^n|\leq 2^{-n}$

b) There is an algorithm printing a sequence (of deg.s and coefficient lists of) $(P_n)\subseteq\mathbb{D}[X]$ with $\|f-P_n\|_\infty\leq 2^{-n}$

c) The real sequence $f(q)$, $q\in\mathbb{D}\cap[0,1]$, is computable $\wedge$ f admits a computable modulus of continuity

Theorem: To approximate $|x-½|$ up error 2^{-n} requires polynomials of degree exponential in n.

Definition: A WHILE+ program computes $r\in\mathbb{R}$ in **polytime** iff, on input n after $\leq$**poly(*n*)** steps, it returns some $a_n\in\mathbb{Z}$ with $|r-a_n/2^n|\leq 2^{-n}$.

Properties of Polytime Functions

Def: Computing $f:\subseteq\mathbb{R}\to\mathbb{R}$ in time $t(n)$ means to print, given $(\underline{a})_m\subseteq\mathbb{Z}$ with $|\underline{x}-\underline{a}_m/2^m|\leq 2^{-m}$ for $\underline{x}\in\mathrm{dom}(f)$, $(\underline{b}_n)\subseteq\mathbb{Z}$ such that $|f(\underline{x})-\underline{b}_n/2^n|\leq 2^{-n}$ within $t(n)$ steps.

Runtime may depend only on output precision n.

Theorem: If $f:[0;1]\to\mathbb{R}$ is computable, then so within bounded time $t(n)$ for some $t:\mathbb{N}\to\mathbb{N}$

Theorem: If $f:\subseteq\mathbb{R}\to\mathbb{R}$ is computable in time $t(n)$, then $\mu(n):=t(n+1)+1$ is a modulus of continuity of f.

Definition: A WHILE+ program computes $r\in\mathbb{R}$ in **polytime** iff, on input n after $\leq$**poly(*n*)** steps, it returns some $a_n\in\mathbb{Z}$ with $|r-a_n/2^n|\leq 2^{-n}$.

Complexity of 1D Maximization

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Fix polytime-comput. $f:[0;1]\to[0;1]$ ($\Rightarrow$continuous)

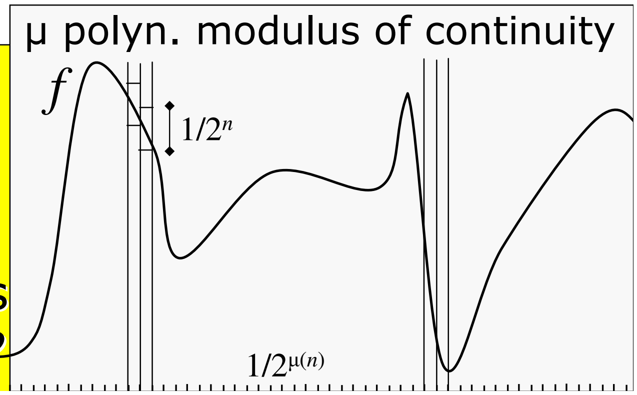
$\mathrm{Max}(f): [0;1] \ni x \to \max\{\, f(t): t\leq x \,\}$

is computable in exponential time

is polytime-computable, provided that $\mathcal{P}=\mathcal{NP}$

if polytime for every (smooth) f, then $\mathcal{P}=\mathcal{NP}$:

Thm [Friedman&Ko'82]
To every $L\in\mathcal{NP}$ exists polytime C^∞ $g_L:[0;1]\to\mathbb{R}$ s.t. $[0;1]\ni x\to\max g_L|_{[0,x]}$ is again polytime iff $L\in\mathcal{P}$



'Max is $\mathcal{NP}$-hard'

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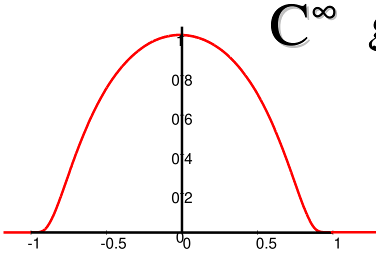
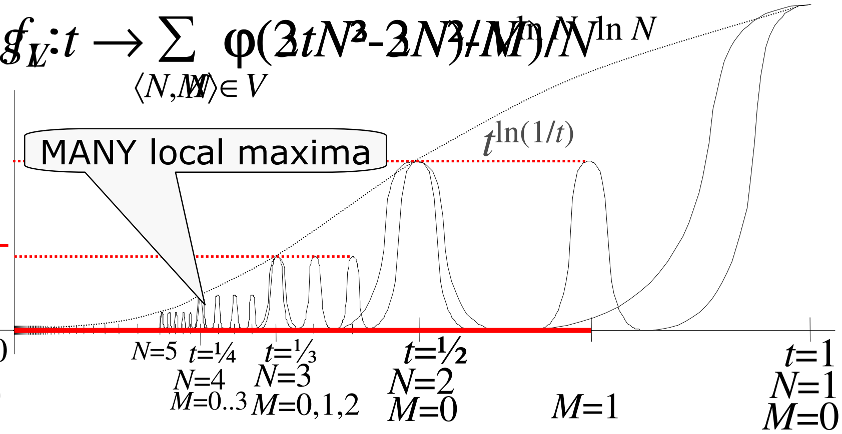
$\mathcal{NP} \ni L = \{\, N\in\mathbb{N} \mid \exists M<N: \langle N,M\rangle\in V \,\}$, $V\in\mathcal{P}$ polytime

C^∞ $g_L: t \to \sum_{\langle N,M\rangle\in V} \varphi(2tN^3-2N^3-M)/N^{\ln N}$



$\varphi(t) = \exp(-t^2/1-t^2)$

C^∞ 'pulse' function
polytime computable



To every $L\in\mathcal{NP}$ there exists a polytime computable C^∞ function $g_L:[0;1]\to\mathbb{R}$ s.t.: $[0;1]\ni x\to\max g_L|_{[0;x]}$ again polytime iff $L\in\mathcal{P}$

Complexity Conjectures in Numerics

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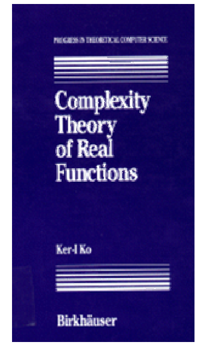
Fix polytime f:[0;1]→[0;1] (⇒ continuous)



- Max: $f \to \mathrm{Max}(f): x \to \max\{\, f(t): t \le x \,\}$
  Max(f) computable in exponential time;
  polyn.time-computable iff $\mathcal{P}=\mathcal{NP}$
- $\int: f \to \int f: (x \to \int_0^x f(t)\,dt)$ **even** for $f \in C^\infty$
  $\int f$ computable in exponential time;
  polyn.time-computable iff $\mathcal{P}=\#\mathcal{P}$

  Polyn.time for <u>analytic</u> $f \in C^\omega$
- odesolve: $C^1([0;1]\times[-1;1]) \ni f \to z$: $\dot{z}(t)=f(t,z)$, $z(0)=0$.
  $\mathcal{PSPACE}$-"complete" [Kawamura 2010]
- Solution to Poisson's Equation
  is classical and $\#\mathcal{P}$-"complete"

  $$\Delta u = f \text{ on } B_2(\mathbf{0},1)$$
  $$u = 0 \text{ on } \partial B_2(\mathbf{0},1)$$

[Kawamura+Steinberg+Z., MSCS 2017

Perspective & Visioin

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- Computing on (σ-) compact metric spaces
- Rigorous Computability and Complexity Theory of *partial* differential equations (PDEs)
  - Why error bound 2^{-n} rather than $1/n$?
  - Why absolute errors rather than relative ?
  - Why inputs only by approximation ?