CS493
Summer 2018, Assignment #1

PROBLEM 1 (1P+2P*):
Recall that decision problem \( X \subseteq \mathbb{N} \) is called reducible to \( Y \subseteq \mathbb{N} \) (written \( X \preceq Y \)) if there exists a total computable function \( f : \mathbb{N} \to \mathbb{N} \) such that, for all \( x \in \mathbb{N} \), it holds: \( x \in X \iff f(x) \in Y \).

a) Prove \( T \preceq E \).
b) Prove \( E \preceq T \).

Here we recall the Totality problem \( T \) and consider the following problem:

\( E \) Given two (finite binary strings encoding) algorithms/WHILE+ programs \( A \) and \( B \), are they equivalent in the sense that, for every \( x \in \mathbb{N} \), \( A \) on input \( x \) eventually terminates iff \( B \) on input \( x \) does (although not necessarily after the same number of steps)?

\( T \) Given an algorithm/WHILE+ program \( A \), does it terminate on all possible inputs \( x \)?

PROBLEM 2 (2P):
Recall the Bachmann–Landau symbols \( \mathcal{O}, \Omega, o, \omega, \Theta \) of asymptotic growth of functions \( f, g : \mathbb{N} \to [1; \infty) \). Then classify the asymptotic growth of the following functions as logarithmic, polynomial, exponential, or in-between: (i) \( \log(n!) \), (ii) \( n^{\log n / \log \log n} \), (iii) \( 2^{(\log n)^2} \).

PROBLEM 3 (2P+1P):

a) Devise a WHILE+ program with one argument \( x \) computing \( 2^x \).
b) Devise a WHILE+ program with argument \( x \) computing the exponential tower \( 2^{2^\cdots^2} \) of height \( x \).

PROBLEM 4 (1P+1P+1P+1P):
Prove these connections between decision problems \( L \subseteq \mathbb{N} \) and discrete functions \( f : \mathbb{N} \to \mathbb{N} \):

a) Each step of a WHILE+ program with variables \( (x_0, \ldots, x_d) = \vec{x} \) can increase \( \ell(\vec{x}) := \max\{\ell(x_0), \ldots, \ell(x_d)\} \) by at most one.
b) If \( f \) can be computed in polynomial time, there exists a \( k \in \mathbb{N} \) with \( \ell(f(x)) \leq O(\ell(x)^k) \) for all \( x \in \mathbb{N} \), where \( \ell(x) = \lceil \log_2(1+x) \rceil \) denotes the binary length of \( x \).
c) If \( f \) can be computed in polynomial time, then the following decision problems lies in \( \mathcal{P} \):

\[
\text{Subgraph}(f) = \{ \langle x, y \rangle : x \in \mathbb{N}, y \leq f(x) \}
\]

for the Pairing Function \( \langle x, y \rangle = x + (x+y) \cdot (x+y+1)/2 \) computable and invertible in polynomial time.
d) If \( \text{Subgraph}(f) \) is decidable in polynomial time and \( \ell(f(x)) \leq O(\ell(x)^k) \) holds for some \( k \) and all \( x \), then \( f \) is computable in polynomial time.