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CS493

Summer 2018, Assignment #1

PROBLEM 1 (1P+2P*)**:**
Recall that decision problem $X \subseteq \mathbb{N}$ is called *reducible* to $Y \subseteq \mathbb{N}$ (written $X \preccurlyeq Y$) if there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in \mathbb{N}$, it holds: $x \in X \Leftrightarrow f(x) \in Y$.

a) Prove $T \preccurlyeq E$.

b) Prove $E \preccurlyeq T$.

Here we recall the Totality problem T and consider the following problem:

E) Given two (finite binary strings encoding) algorithms/WHILE+ programs $\mathcal{A}$ and $\mathcal{B}$, are they equivalent in the sense that, for every $x \in \mathbb{N}$, $\mathcal{A}$ on input x eventually terminates iff $\mathcal{B}$ on input x does (although not necessarily after the same number of steps)?

T) Given an algorithm/WHILE+ program $\mathcal{A}$, does it terminate on *all* possible inputs x?

PROBLEM 2 (2P)**:**
Recall the Bachmann–Landau symbols $\mathcal{O}$, Ω, o, ω, Θ of asymptotic growth of functions $f, g : \mathbb{N} \to [1;\infty)$.Then classify the asymptotic growth of the following functions as logarithmic, polynomial, exponential, or in-between: (i) $\log(n!)$, (ii) $n^{\log\log n/\log n}$, (iii) $2^{(\log n)^2}$.

PROBLEM 3 (2P+1P)**:**

a) Devise a `WHILE+` program with one argument x computing 2^x.

b) Devise a `WHILE+` program with argument x computing the exponential tower $2^{2^{\cdot^{\cdot^{2}}}}$ of height x

PROBLEM 4 (1P+1P+1P+1P)**:**
Prove these connections between decision problems $L \subseteq \mathbb{N}$ and discrete functions $f : \mathbb{N} \to \mathbb{N}$:

a) Each step of a WHILE+ program with variables $(x_0, \ldots, x_d) = \vec{x}$ can increase $\ell(\vec{x}) := \max\{\ell(x_0), \ldots, \ell(x_d)\}$ by at most one.

b) If f can be computed in polynomial time, there exists a $k \in \mathbb{N}$ with $\ell\big(f(x)\big) \leq \mathcal{O}\big(\ell(x)^k\big)$ for all $x \in \mathbb{N}$, where $\ell(x) = \lceil \log_2(1+x) \rceil$ denotes the binary length of x.

c) If f can be computed in polynomial time, then the following decision problems lies in $\mathcal{P}$:

$$\text{Subgraph}(f) \;=\; \big\{ \langle x, y \rangle : x \in \mathbb{N}, y \leq f(x) \big\}$$

for the Pairing Function $\langle x, y \rangle \;=\; x \,+\, (x+y) \cdot (x+y+1)/2$ computable and invertible in polynomial time.

d) If Subgraph(f) is decidable in polynomial time and $\ell\big(f(x)\big) \leq \mathcal{O}\big(\ell(x)^k\big)$ holds for some k and all x, then f is computable in polynomial time.