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CS493

Summer 2018, Assignment #2

PROBLEM 5 (1+2+1+1+1+2+1+1P)**:**
Write $\mathbb{D}_n := \mathbb{Z}/2^n \subseteq \mathbb{Q}$ and $\mathbb{D} := \bigcup_n \mathbb{D}_n$ for the set of dyadic rationals. Let $H \subseteq \mathbb{N}$ denote the semi-decidable but undecidable Halting Problem. Abbreviate $\{0,1,2,\ldots,\infty\} \ni t(\langle \mathcal{A}, x \rangle) :=$ #steps algorithm $\mathcal{A}$ makes on input x. Prove:

a) Every dyadic rational has precisely two binary expansions,
all other real numbers have precisely one.

b) If $a, b \in \mathbb{R}$ are computable, then so are $a+b$ and $a \cdot b$ and $1/a$ ($a \neq 0$) and $\sqrt{a}$ ($a > 0$).

c) Fix $p \in \mathbb{R}[X]$.
Then p's coefficients are computable iff $p(x)$ is computable for every computable $x \in \mathbb{R}$.

d) Every computable $f : [0;1] \to [-1;1]$ with $f(0) < 0 < f(1)$ has a computable root.

e) Every algebraic number is computable.

f) If $x \in \mathbb{R}$ is computable, then so is $\exp(x)$.

g) Specker's Sequence $s_j := \sum_{\substack{m \leq j \\ t(m) \leq j}} 2^{-m} \in \mathbb{D}_j$ is computable, but its limit is not.

h) The sequence $r_j := 2^{-t(j)}$ is computable, yet $\{j : r_j \neq 0\} = H$.