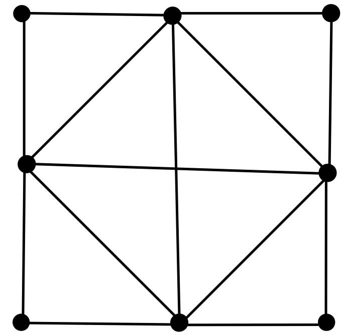
§8 Approximation

Lemma: i) The vertices of any *maximal* matching constitute a vertex cover.

ii) The latter is at most twice as large as a *minimal* one.



Example a) maximal matching (size 2)
b) largest matching (size 4)

Theorem: Greedy algorithm for maximal matching yields factor-2 approximation to **min**VC in time $O(|E|)$.

A *matching* in G is a subset $M \subseteq E$ wherein no two edges share a vertex.

VC := { $\langle G,k \rangle$ | G has a vertex cover of size $\leq k$ }

Approximating metric TSP

MTSP = { $\langle G,\underline{w},k \rangle$ | G with metric edge weights $\underline{w}: V\times V \to \mathbb{N}$ admits a Hamiltonian circuit of weighted length $\leq k$ }

Input: $\underline{w}: V\times V \to \mathbb{N}$ edge weights symmetric and s.t. triangle inequality holds: $w(a,c) \leq w(a,b) + w(b,c)$

Sought: Tour (permutation π of V) of least weight

Decision problem **MTSP** still $\mathcal{NP}$-complete

[Christofides'76]

Polytime approximating **minMTSP** up to factor 2:

1. Compute minimum spanning tree T of (G,w).
2. Traverse T depth-first pre-order

ETSP with $V \subseteq \mathbb{R}^d$, $w(\underline{a},\underline{b}) = \|\underline{a}-\underline{b}\|_2$ $\mathcal{NP}$-hard, but **in** $\mathcal{NP}$ **?**

Proof of Approximation Ratio

<u>*w*</u> weights with 3-inequality, *T* is MST traversed in-order

Let F denote the sequence of edges pursued in-order, C the tour thus obtained, C^* an optimal tour.

For edges $e_1,\ldots e_k$ abbreviate $L(e_1,\ldots e_k) := w(e_1)+\ldots+w(e_k)$

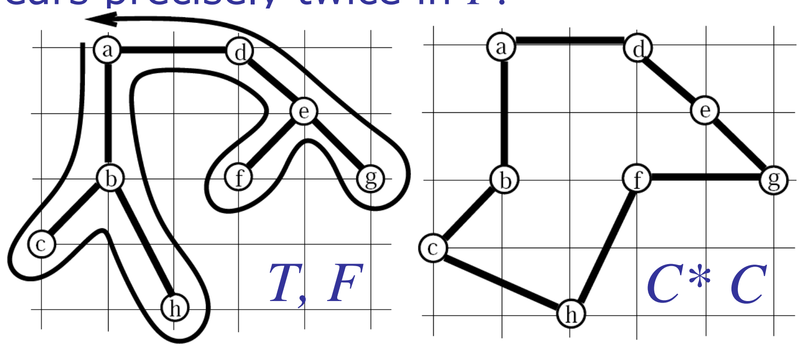
(i) $L(T) \leq L(C^*)$, because removing any edge from C^* yields a spanning tree of cost $\leq L(C^*)$

Every edge of T appears precisely twice in F:

(ii) $L(F) = 2\cdot L(T)$

(iii) $L(C) \leq L(F)$ because 3-inequal.

$\Rightarrow L(C) \leq L(F) = 2\cdot L(T) \leq 2\cdot L(C^*)$



Approximation Schemes

Input: n packets, values $v_1,\ldots v_n \in \mathbb{N}$ and weights $w_1,\ldots w_n \in \mathbb{N}$ and weight/value bounds W,V

Question: Is there a subset $S \subseteq \{1,\ldots,n\}$ s.t. values $\sum_{p\in S} v_p \geq V$ subject to weight bound $\sum_{p\in S} w_p \leq W$

Now: Find S' s.t. $\sum_{p\in S'} w_p \leq W$ and $\sum_{p\in S'} v_p \geq V\cdot(1-\varepsilon)$

Or: Find S'' s.t. $\sum_{p\in S''} w_p \leq W\cdot(1+\varepsilon)$ and $\sum_{p\in S''} v_p \geq V$

Algorithm: guaranteed approximation ratio $1\pm\varepsilon$

Discrete optimization → decision often $\mathcal{NP}$-hard

Try approximating maxim./minim. up to relative error

Dynamic Programming: *Knapsack*

KAIST CS500 M. Ziegler

For $S \subseteq \{1,\ldots,n\}$ write $w(S)=\sum_{p\in S} w_p$ and $v(S)=\sum_{p\in S} v_p$

Goal: Given W, determine $V := \max\{\, v(S) : w(S) \leq W \,\}$

Consider $T(v,m) := \min\{\, w(S) : S\subseteq\{1,\ldots m\},\ v(S)\geq v \,\}$

Note: i) $T(0,n) \leq T(1,n) \leq \ldots \leq T(V,n) \leq W < T(V+1,n)$

ii) $V = \max\{\, v : T(v,n) \leq W \,\}$

iii) $T(v,m) = 0$ for $v \leq 0$

iv) $T(v,0) = \infty$ for $v>0$

w.l.o.g. $0<w_p\leq W$, $0<v_p \leq V$

v) $T(v,m) = \min\{\, T(v,m-1),\ w_m + T(v-v_m, m-1) \,\}$

$v \backslash m$	0	1	…	n
0	0	0	0	0
1	∞			
2	∞			
⋮	∞			T

runtime $\mathrm{poly}(n+V)$

FPTAS for *Knapsack*

KAIST CS500 M. Ziegler

Scaling Lemma a) For $0\leq \underline{v}' \leq \underline{v}$, $V(\underline{v}') \leq V(\underline{v})$

b) and for $\underline{v} \leq \underline{d} \leq (k,\ldots k)$: $V(\underline{v}-\underline{d}) \geq V(\underline{v}) - n\cdot k$

c) Also, $V(k\cdot \underline{v}) = k \cdot V(\underline{v})$

$v-k < \lfloor v/k \rfloor \cdot k \leq v$

Scaling Method: Fix $k\in\mathbb{N}$ and let $v_p' := \lfloor v_p/k \rfloor$

Compute $V' := k \cdot V(v_1',\ldots,v_n')$ in time $\mathrm{poly}(n+V/k)$. So

$V' = V(\lfloor \underline{v}/k \rfloor \cdot k) \geq V(\underline{v}-k\cdot\underline{1}) \geq V-n\cdot k = V\cdot(1-n\cdot k/V) \overset{!}{\geq} V\cdot(1-\varepsilon)$

for $k :\approx \lfloor \varepsilon\cdot\sum_p v_p/n^2 \rfloor \leq \varepsilon\cdot V/n$ $\quad$ $0 < v_p \leq V \Rightarrow V \leq \sum_p v_p \leq nV$ $\quad$ $V/k \leq O(n^2/\varepsilon+1)$

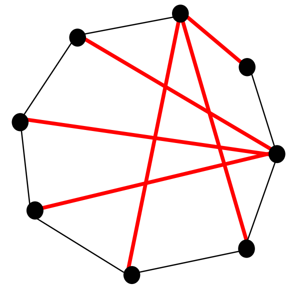
Theorem: For every given $\varepsilon>0$, can approximate Knapsack up to error $1-\varepsilon$ in time polynom. in $n+1/\varepsilon$

$V(v_1,\ldots v_n) := \max\{\, \sum_{p\in S} v_p : S\subseteq\{1..n\},\ \sum_{p\in S} w_p \leq W \,\}$

Limits of Approximation

Theorem: No polynom.-time algorithm can approximate the general TSP up to some constant unless $\mathcal{P}=\mathcal{NP}$.

Proof: Suppose $\mathcal{A}$ approximates TSP up to factor $c\in\mathbb{N}$.
Turn $\mathcal{A}$ into algorithm $\mathcal{B}$ for HC:



Algorithm $\mathcal{B}$, input graph $G=(V,E)$, $n:=|V|$.
Define $w(u,v) := 1$ for $\{u,v\}\in E$;
$w(u,v) := n\cdot c$ for $\{u,v\}\notin E$.

No triangle-inequality…

$\langle G\rangle\in$ **HC** $\Rightarrow$ w contains Hamiltonian cycle of weight n
$\Rightarrow$ algorithm $\mathcal{A}$ finds some of weight $\le n\cdot c$

$\langle G\rangle\notin$ **HC** $\Rightarrow$ any Hamiltonian cycle has weight $> n\cdot c$

HC := { $\langle G\rangle$ | graph G contains a Hamiltonian cycle }
TSP := { $\langle G,w,k\rangle$ | (G,w) contains a Hamiltonian cycle of weight $\le k$ }

In-/Approximability

Can approximate in polynomial time:

- **Knapsack** up to error 1-ε for any fixed ε>0
- **VertexCover** up to error 1+1=**2**
- **metricTSP** up to error **2**
- **Clique** up to error n, trivially

Unless $\mathcal{P}=\mathcal{NP}$, can<u>not</u> approximate

- (general) **TSP** up any to constant error
- **CLIQUE** up to error $O(n^{1-\varepsilon})$ **[Johan Håstad'96]**