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Issued on May 12, 2018
Solutions due: May 23, 2018

CS500

Spring 2018, Assignment #5

PROBLEM 11 (2+3P) **:**
Fix $0 < D \in \mathbb{N}$ and abbreviate $p_x := \left(1-\frac{1}{D}\right)^{D-1-x} \cdot D^{D-1} / \left(D^D - (D-1)^D\right)$ for $x = 0,1,2,\ldots,D-1$, $p_x := 0$ for $x \geq D$.

a) Prove that $\sum_x p_x = 1$.

b) Prove that FIFO on k pages is k-competitive.

Definition: A Boolean term Φ (over variables $x_1,\ldots,x_n$) is in *4-conjunctive normal form* (4CNF) if it is a conjunction of disjunctions of four *literals* each:

$$\Phi \quad = \quad (\ell_1 \vee \ell_2 \vee \ell_3 \vee \ell_4) \wedge (\ell_5 \vee \ell_6 \vee \ell_7 \vee \ell_8) \wedge \ldots \wedge (\ell_{4m-3} \vee \ell_{4m-2} \vee \ell_{4m-1} \vee \ell_{4m}) \qquad (1)$$

where each ℓ_j is a *literal*, that is, some variable x_i or its negation $\neg x_i$.

PROBLEM 12 (4+3+3P) **:**

a) Design and analyze a polynomial-time algorithm deciding 2SAT:
the question of whether a given Boolean term in 2CNF has a satisfying assignment.

b) Prove that Φ from Equation (1) in variables $x_1,\ldots,x_n$ admits a satisfying assignment iff the following $\widetilde{\Phi}$ in 3CNF and variables $x_1,\ldots,x_n,y_1,\ldots,y_m$ does:

$$\widetilde{\Phi} \quad = \quad (\ell_1 \vee \ell_2 \vee y_1) \wedge (\neg y_1 \vee \ell_3 \vee \ell_4) \wedge (\ell_5 \vee \ell_6 \vee y_2) \wedge (\neg y_2 \vee \ell_7 \vee \ell_8) \wedge \\ \wedge \ldots \wedge (\ell_{4m-3} \vee \ell_{4m-2} \vee y_m) \wedge (\neg y_m \vee \ell_{4m-1} \vee \ell_{4m})$$

c) Recall the decision problem VC (Vertex Cover) and formalize SC (Set Cover):

> Given $n,N,K \in \mathbb{N}$ and a collection of subsets $S_1,\ldots,S_N \subseteq \{1,\ldots,n\}$,
> do there exist $i_1,\ldots,i_K \in \{1,\ldots,N\}$ such that $\bigcup_{k=1}^{K} S_{i_k} = \{1,\ldots,n\}$?

Establish a polynomial-time reduction from VC to SC.

PROBLEM 13 (2+1+2+2P) **:**

a) Draw all real solutions to the following system of bivariate polynomial inequalities:

$$x \leq 4 \quad \wedge \quad x \geq -4 \quad \wedge \quad -4 \leq y \leq 4 \quad \wedge \\ \wedge \quad x^2 + 9\cdot(y+2)^2 > 9 \quad \wedge \quad (x+2)^2 + (y-2)^2 > 1 \quad \wedge \quad (x-2)^2 + (y-2)^2 > 1$$

b) Find all real solutions to this system of n-variate polynomial in-/equalities:

$$x_1^2 = 2 \quad \wedge \quad x_2^2 = x_1 \quad \wedge \quad x_3^2 = x_2 \quad \wedge \ldots \wedge \quad x_n^2 = x_{n-1}$$

And all real solutions to this system: $x^2 = -1$.

c) Formalize the following decision problem:

> Given a system of multivariate polynomial inequalities with integer coefficients, does it have a real solution?

How do you encode the coefficients, how the degrees etc?

d) Establish a polynom.time reduction from Boolean Satisfiability SAT to the problem from (c).

e*) Does the problem from (c) belong to $\mathcal{NP}$?
Argue informally why/why not. What would be the witness?