4 Computational Complexity

- Model of computation with (bit) cost: **WHILE**
- Complexity of Arithmetic
- Complexity classes \( \mathcal{P}, \mathcal{NP}, \mathcal{PSPACE}, \mathcal{EXP} \)
- and their inclusion relations
- Encoding graphs/non-integer data
- Example problems: 3COL, EC, HC, VC, ILP, IS, Clique

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**Model of Computational Cost**

**WHILE** takes expon. time to add two \( n \)-bit integers

Now **WHILE+** programs: Input \( x_1 \in \mathbb{N} \), output \( x_0 \in \mathbb{N} \)

\[
\begin{align*}
x_j &:= 0 \mid x_j := 1 \mid x_j := x_i + x_k \mid x_j := x_i - x_k \\
x_j &:= x_i \div 2
\end{align*}
\]

**P;P | WHILE \ x_i \ DO \ P \ END**

**Definitions:**
- **Binary length** of \( x \in \mathbb{N} \): \( \ell(x) = \lceil \log_2(1+x) \rceil \)
- **Time** of a **WHILE+** program \( P \) on input \( \underline{x}=(x_1, \ldots, x_k) \)
- **Space** (=memory) used: \( \max_t \ell(x) := \ell(x_1) + \ldots + \ell(x_k) \)
- **Asymptotic** time/space \( t(n)/s(n) \):
  - worst-case over all inputs \( \underline{x} \) with \( \ell(x)<n \)
- Recall pairing function \( \langle x,y \rangle := x + (x+y) \cdot (x+y+1)/2 \)
Complexity of Arithmetic

WHILE takes expon. time to add two $n$-bit integers
Now WHILE+ programs: Input $x_1 \in \mathbb{N}$, output $x_0 \in \mathbb{N}$

\[
x_j := 0 \mid x_j := 1 \mid x_j := x_i + x_k \mid x_j := x_i - x_k \mid x_j := x_i \div 2 \mid P; P \mid \text{WHILE } x_i \text{ DO } P \text{ END}
\]

Multiplication by repeated addition: expon. time
Long multiplication: linear time
Long division: linear time
Un-/pairing: linear time

- asymptotic time/space $t(n)/s(n)$:
  - worst-case over all inputs $x$ with $\ell(x) < n$
- Recall pairing function $\langle x, y \rangle := x + (x+y) \cdot (x+y+1)/2$

Preliminaries: Graphs and Coding

A directed graph $G=(V,E)$ is a finite set $V$ of vertices and a set $E \subseteq V \times V$ of edges.

- Call $G$ undirected if it holds $(u,v) \in E \iff (v,u) \in E$
- Sometimes $c: E \rightarrow \mathbb{N}$ assigning weights to edges.

For input to a WHILE+ program:
- Represent $(G,c)$ as adjacency matrix $A \in \mathbb{N}^{V \times V}$
  - $A[u,v] := c(i,j)$ for $(u,v) \in E$
  - $A[u,v] := \infty$ for $(u,v) \notin E$
- Undirected case: only upper triangular matrix.
- Encoding $\langle G,c \rangle \in \mathbb{N}$ has $|V| \leq |\langle G,c \rangle| \leq O(|V|^2 \cdot \log |c|_\infty)$
Some Complexity Classes

**Definition:** a) A WHILE+ program computes the function \( f: \mathbb{N} \to \mathbb{N} \) if on input \( x \) it prints \( f(x) \) and terminates in time \( t(n) / \) space \( s(n) \), \( n := \ell(x) \)

**Polynom. growth:** \( \exists k \ t(n) \leq O(n^k) \); **exponential:** \( 2^{O(n^k)} \)

**Def:** For decision problems \( L \subseteq \mathbb{N} \) or \( L \subseteq \{0,1\}^* \)

- \( \mathcal{P} = \{ L \text{ decidable in polynomial time} \} \)
- \( \mathcal{NP} = \{ L \text{ verifiable in polynomial time} \} \), i.e.

\[
L = \{ x \in \mathbb{N} : \exists y \in \mathbb{N}, \ell(y) \leq \text{poly}(\ell(x)), \langle x, y \rangle \in V \}, \quad V \in \mathcal{P}
\]

- \( \mathcal{PSPACE} = \{ L \text{ decidable in polynomial space} \} \)
- \( \mathcal{EXP} = \{ L \text{ decidable in exponential time} \} \)

**Theorem:** \( \mathcal{P} \subseteq \mathcal{NP} \subseteq \mathcal{PSPACE} \subseteq \mathcal{EXP} \)

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**Example Problem (0)**

**Def:** A 3-coloring of \( G=(V,E) \) is a mapping \( \gamma: V \to \{R,G,B\} \) s.t. \( \gamma(u) \neq \gamma(v) \) for every \( (u,v) \in E \).

**Examples:**

a) The Petersen Graph admits a 3-coloring.

b) This graph, too.

c) This one still.

d) But not this one.

\[
x = \langle G \rangle, \quad y = \langle \gamma(1), \ldots, \gamma(|V|) \rangle
\]

**3COL=** \{ \( \langle G \rangle \mid G \text{ admits a 3-coloring} \} \in \mathcal{NP}

\( \mathcal{NP} \ni \{ x \in \mathbb{N} : \exists y, \ell(y) \leq \text{poly}(\ell(x)), \langle x, y \rangle \in V \}, \quad V \in \mathcal{P} \)
Example Problems (I)

In an undirected graph $G$, Eulerian cycle traverses each edge precisely once;
Hamiltonian cycle visits each vertex precisely once.

G allowing a Eulerian cycle is connected and
has an even number of edges incident to each vertex

**Theorem:** Conversely every connected graph
with an even number of edges incident to each vertex
admits a Eulerian cycle.

$$\text{EC} := \{ \langle G \rangle \mid G \text{ has a Eulerian cycle} \} \quad \mathcal{P}$$

$$\text{HC} := \{ \langle G \rangle \mid G \text{ has Hamiltonian cycle} \} \quad \mathcal{NP}$$

Example Problems (II)

- **Eulerian (EC) vs. Hamiltonian Cycle (HC)**
- **(Minimum) Edge Cover**
  "To graph $G$, find a smallest subset $F$ of edges
  s.t. any vertex $v$ is adjacent to at least one $e \in F."$
- **vs. Vertex Cover (VC)**
- **CLIQUE = $\{ \langle G,k \rangle \mid G$ contains a $k$-clique $\}**
- **IS= $\{ \langle G,k \rangle : G$ has $k$ pairwise non-adjacent vertices $\}**
- **Integer Linear Programming $\mathcal{NP}$**

$$\text{ILP} = \{ \langle A,b \rangle : A \in \mathbb{Z}^{n \times m}, b \in \mathbb{Z}^m, \exists x \in \mathbb{Z}^n: A \cdot x = b \}$$

$$\mathcal{NP} \ni \{ x \in \mathbb{N} : \exists y, \ell(y) \leq \text{poly}(\ell(x)), \langle x,y \rangle \in V \}, \quad V \in \mathcal{P}$$
**Example Problems (III)**

**Def:** A Boolean term \( \Phi(Y_1,\ldots,Y_n) \) is composed from variables \( Y_1,\ldots,Y_n \), constants 0 and 1, and operations \( \lor, \land, \neg \).

\[ \Phi \text{ in 3-CNF if } \Phi = \bigwedge (\neg y_i \lor (\neg y_j \lor (\neg y_k)) \]  

**Examples:**
- \( 0 \)
- \( (\neg x \lor y) \land (x \lor \neg y) \)
- \( (\neg x \lor y) \land (x \lor \neg z) \land (z \lor \neg y) \land \neg (\neg y) \)

**EVAL:** Given \( \langle \Phi(Y_1,\ldots,Y_n) \rangle \) and \( y_1,\ldots,y_n \in \{0,1\} \), does \( \Phi(y_1,\ldots,y_n) \) evaluate to 1? \( \in \mathcal{P} \)

\([k]-\text{SAT} \): Given \( \Phi(Y_1,\ldots,Y_n) \) [in 3-CNF], does it hold \( \exists y_1,\ldots,y_n \in \{0,1\} : \Phi(y_1,\ldots,y_n)=1 \)?

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**Non-Deterministic WHILE+**

**Theorem:** \( L \subseteq \mathbb{N} \) is accepted by a non-deterministic polynomial-time WHILE+ program iff \( L \in \mathcal{NP} \).

\[
x_j := 0 \mid x_j := 1 \mid x_j := x_i + x_k \mid x_j := x_i - x_k \mid x_j := x_i \div 2 \mid \text{guess } x_j \mid P;P \mid \text{WHILE } x_i \text{ DO } P \text{ END}
\]

**Definition:** A non-deterministic WHILE+ program may (repeatedly) guess a bit (0/1).

- Its runtime is \( \leq t(n) \) if it makes no more than \( t(\ell(x_1)) \) steps, regardless of the guesses.
- It accepts input \( x_1 \) if there exists some choice of guessed values such as to return \( x_0=1 \).
- It rejects \( x_1 \) if no choice of guesses returns \( x_0=1 \).