Representation Theory of Compact Metric Spaces and Computational Complexity of Continuous Data [1]

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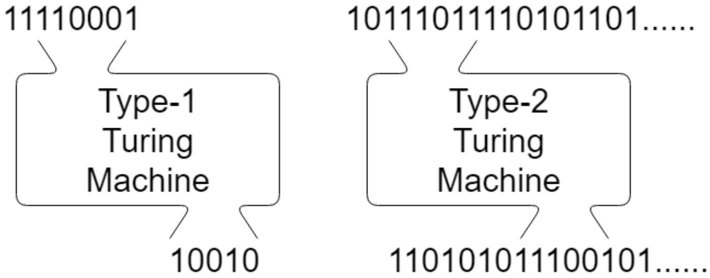
Type-2 Computation



- Input written infinitely
- 1st output symbol, 2nd output symbol, 3rd, 4th ...... forever
- Output cannot be re-written
- Works on forever (never terminates)

Type-2 Computation and Time Cost

Efficiency of an algorithm is measured by a function $t : \mathbb{N} \to \mathbb{N}$

Type-1

input length $n \mapsto$ number of steps until termination

Type-2

output precision $n \mapsto$ number of steps until nth output symbol

By the way, why do we need type-2 computation at all?

Representation

Computational problems we are interested in

- $n \mapsto n$th prime number
- graph $G \mapsto G$ has Hamiltonian path?
- propositional logic formula $\mapsto$ number of satisfying assignments
- a program $\mapsto$ does it halt?

Turing machine model

$$\{0,1\}^* \to \{0,1\}^*$$

How to define computation on $\mathbb{N}$, formulas, graphs?

Representation (type-1)

Def: *Representation* of X

Partial surjective mapping

$$\zeta :\subseteq \{0,1\}^* \twoheadrightarrow X$$

Computation of f defined by *realizer* F

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \uparrow & & \uparrow \\ \{0,1\}^* & \xrightarrow{F} & \{0,1\}^* \end{array}$$

Def: f is *computable*

There exists a type-1 TM computing F such that the diagram commutes

Representations of Countable Sets

- Encoding mathematical objects into 1's and 0's
- Finite length bit sequence

Binary Representation of $\mathbb{N}$

$$\{0,1\}^* \twoheadrightarrow \mathbb{N}$$

$$1101 \mapsto 13$$

Representation of $\mathbb{Q}$

$$\{0,1\}^* \twoheadrightarrow \mathbb{Q}$$

Encodes numerator/denominator separately

- $\{0,1\}^*$ is countable
- Can only encode countable sets!

Representations

- Cantor space $\{0,1\}^{\mathbb{N}}$
- Infinite length bit sequence

Binary Representation of $[0,1]$

$$\{0,1\}^{\mathbb{N}} \twoheadrightarrow [0,1]$$

$$1111\cdots \mapsto 0.1111\cdots_2 = 1$$

Def: *Representation* of Space X

Partial surjective mapping

$$\xi \subseteq: \{0,1\}^{\mathbb{N}} \twoheadrightarrow X$$

Representations

Tertiary Representation

$$\{0,1,2\}^{\mathbb{N}} \twoheadrightarrow [0,1]$$

Signed-Bit Representation

$$\{-1,0,1\}^{\mathbb{N}} \twoheadrightarrow [0,1]$$

Dyadic Representation

$$x \in [0,1]$$

A sequence $a_0, a_1, a_2...$ of dyadic rationals converging fast to x

$$(a_n)_n \mapsto x$$

Representation (type-2)

Representation := Encoding into 1's and 0's

Def: *Representation* of X

Partial surjective mapping

$$\zeta :\subseteq \{0,1\}^{\mathbb{N}} \twoheadrightarrow X$$

Computation of f defined by *realizer* F

$$\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & Y \\
\uparrow & & \uparrow \\
\{0,1\}^{\mathbb{N}} & \xrightarrow{\;F\;} & \{0,1\}^{\mathbb{N}}
\end{array}$$

Def: f is *computable*

There exists a type-2 TM computing F such that the diagram commutes

Which representation to choose?

Type-1

- Can convert back and forth between different representations
- Unary vs binary representation of $\mathbb{N}$
- Do not matter/trivial

Type-2

- Choose binary representation of $\mathbb{R}$: the map $x \mapsto 3x$ is rendered incomputable [Turing, 1936]
- Cauchy representation vs signed bit representation: equivalent in computability

Problem with binary representation

Theorem [Turing, 1936]

$f(x) = 3x$ is incomputable with binary representation

Proof Idea.

- Suppose a TM M exists that computes $f(x) = 3x$
- Feed M with input 0.3333......
- M should print either 0.9999...... or 1.0000......
- But if the input were 0.3333......0000......, then M should print 0.9999....
- But if the input were 0.3333......9999......, then M should print 1.0000......
- How could M determine the first output symbol?

Computability of Real Functions

Most common choice: Cauchy representation

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\;f\;} & \mathbb{R} \\
\scriptstyle\mathit{Cauchy}\uparrow & & \scriptstyle\mathit{Cauchy}\uparrow \\
\{0,1\}^{\mathbb{N}} & \xrightarrow{\;F\;} & \{0,1\}^{\mathbb{N}}
\end{array}
$$

Definition

f is *computable* iff there exists a type-2 TM that computes realizer F.

Computability of Real Functions

Definition, in other words

$f : \mathbb{R} \to \mathbb{R}$ is *computable* iff there exists a type-2 TM that

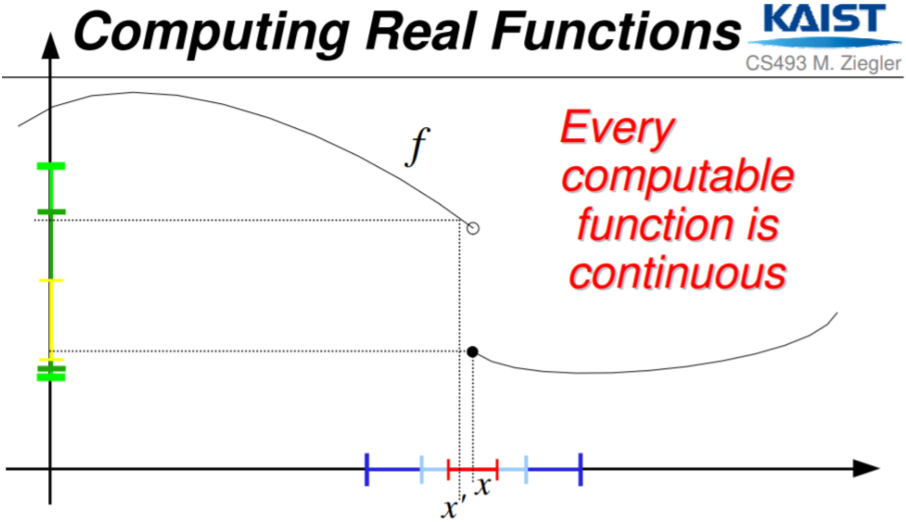
- given an infinite sequence $a_0, a_1, \cdots \subseteq \mathbb{Q}^2$ converging fast to x
- outputs an infinite sequence $b_0, b_1, \cdots \subseteq \mathbb{Q}^2$ converging fast to $f(x)$

Theorem

Computable functions are continuous
Discontinous functions are incomputable

Proof Idea. How could a machine determine precise output when the given input is only an approximation?

Discontinuous Functions are Incomputable



Continuity and Computability

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle Cauchy}\uparrow & & {\scriptstyle Cauchy}\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{F} & \{0,1\}^{\mathbb{N}} \end{array}$$

Theorem

The followings are equivalent

- f continuous
- F oracle computable
- F continuous

We will be concerned mainly about continuity

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Generalization [Kreitz, Weirauch, 1985]

X, Y: topological spaces with T_0 and second countability
Under the criterion: **admissible representations**

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{admissible}\uparrow & & \text{admissible}\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{F} & \{0,1\}^{\mathbb{N}} \end{array}$$

Main Theorem of Computability for Continuous Data

f continuous $\Leftrightarrow \exists F$ continuous

- Cauchy representation is admissible
- Binary representation is not

Admissibility [Kreitz, Weirauch, 1985]

Distinguish reasonable encodings from bad ones

ζ reduces to ξ

$$\begin{array}{ccc} \{0,1\}^{\mathbb{N}} & \xrightarrow{\ \xi\ } & X \\ F\uparrow & \overset{\zeta}{\nearrow} & \\ \{0,1\}^{\mathbb{N}} & & \end{array}$$

$\exists F$ such that $\zeta = \xi \circ F$

Def: Admissibility (similar to completeness)

A representation ξ is *admissible* iff

- continuous
- any continuous representation ζ is continuosly reducible to ξ

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle Cauchy}\uparrow & & {\scriptstyle Cauchy}\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{F} & \{0,1\}^{\mathbb{N}} \end{array}$$

Real Numbers $\mathbb{R}$

f continuous $\Leftrightarrow$ F continuous

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle admissible}\uparrow & & {\scriptstyle admissible}\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{F} & \{0,1\}^{\mathbb{N}} \end{array}$$

Topological Spaces X, Y

f continuous $\Leftrightarrow$ F continuous

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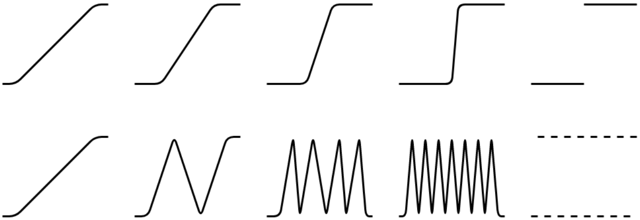
Continuity to modulus of continuity



- Qualitative to quantitative
- Computability to complexity: how **difficult** is a problem?
- Continuity to modulus of continuity: how **discontinuous** is a function?

Continuity

We consider only uniform continuity

Definition

f is *uniformly continuous* iff for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

The map $\epsilon \mapsto \delta$ measures **discontinuity** of f

- Lipshitz continuity $\Leftrightarrow$ Linear map $\epsilon \mapsto \delta$
- Hölder continuity $\Leftrightarrow$ Polynomial map $\epsilon \mapsto \delta$

Modulus of Continuity

Metric spaces $(X, d), (Y, e)$

Def: *Modulus of Continuity* $\mu : \mathbb{N} \to \mathbb{N}$ of $f : X \to Y$

For any $a, b \in X$

$$d(a, b) \le 2^{-\mu(n)} \Rightarrow e(f(a), f(b)) \le 2^{-n}$$

The same as previous one, measured in terms of exponents

- Lipschitz continuity $\Leftrightarrow \mu = n + O(1)$
- Hölder continuity $\Leftrightarrow \mu = O(n)$

Theorem

Real function polytime computable $\Rightarrow$ polynomial modulus

Continuity : Modulus of continuity = Computability : Complexity

Modulus of continuity of a representation

Metric of Cantor Space

$a, b \in \{0,1\}^{\mathbb{N}}$

$$d(a,b) := 2^{-\min\{j \mid a_i \neq b_i\}}$$

A representation of X

$$\zeta : \{0,1\}^{\mathbb{N}} \twoheadrightarrow X$$

- Modulus $\mu : n \mapsto$ number of bits needed to ensure precision 2^{-n}
- Smaller modulus continuity, more efficient
- How efficient can a representation be?
- $[0,1]$ vs $C([0,1],[0,1])$

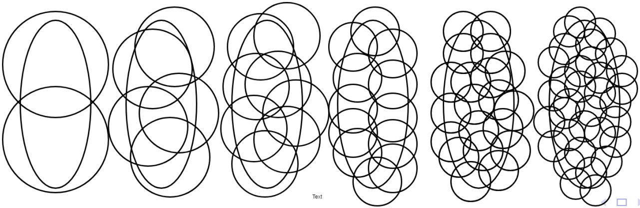
Entropy

- Compact metric space X
- How big/complicated is X?
- How many closed balls needed to cover X?

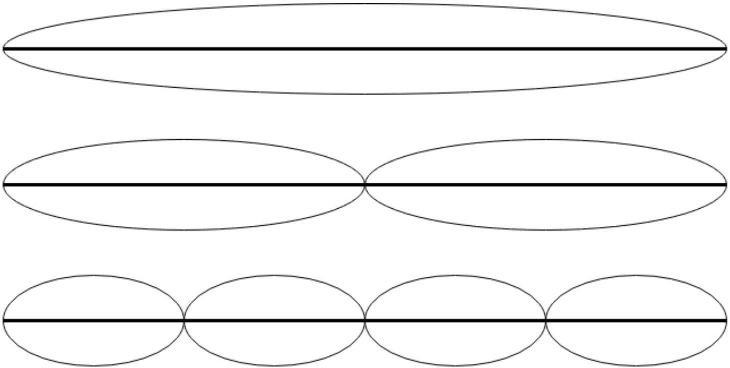
Def: *Entropy* η of X [Kolmogorov, 1959]

$$\eta : \mathbb{N} \to \mathbb{N}$$

Precision parameter $n \mapsto \log_2$ of minimum number of balls of radius 2^{-n} needed to cover X
Can cover X by $2^{\eta(n)}$ balls but not $2^{\eta(n)-1}$



Entropy of $[0, 1]$



- Half the radius, Twice more balls
- $\eta(n) = n + O(1)$
- $[0, 1]^2 \Rightarrow \eta(n) = 2n + O(1)$
- $[0, 1]^3 \Rightarrow \eta(n) = 3n + O(1)$

Entropy

- Compact metric space X
- How complex is X?
- $[0, 1]$ has entropy n
- $[0, 1]^2$ has entropy $2n$
- $[0, 1]^3$ has entropy $3n$
- $[0, 1]^{\mathbb{N}}$ has entrotpy $\Theta(n^2)$
- $Lip_1([0, 1], [0, 1])$ has entropy $\Theta(2^n)$

Lower Bound of Modulus of Continuity

Big/complicated spaces are difficult to encode

- Compact metric space X
- Any of X's representation $\xi : \{0,1\} \twoheadrightarrow X$

Observation [Steinberg, 2015]

entropy η of $X \leq$ modulus of continuity μ of ξ

$$\eta(n) \leq \mu(n)$$

A representation ξ is considered optimal if $\mu \leq O(\eta)$

Observation (Steinberg, 2015)

entropy η of $X \leq$ modulus of continuity μ of ξ

Entropy of $[0, 1]$

$$\eta(n) = n + O(1)$$

Binary Representation $\mu = \Theta(\eta)$ (Optimal)

$$\{0, 1\}^{\mathbb{N}} \twoheadrightarrow [0, 1]$$

Cauchy Representation $\mu = \Theta(\eta^2)$ (Not optimal)

$$(a_n)_n \mapsto x \in [0, 1]$$

Signed-Bit Representation $\mu = \Theta(\eta)$ (Optimal)

$$\{-1, 0, 1\}^{\mathbb{N}} \twoheadrightarrow [0, 1]$$

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Our Refinement

- Qualitative $\Rightarrow$ Quantitative
- Computability $\Rightarrow$ Complexity
- Topological spaces[KW85] $\Rightarrow$ Compact metric spaces
- Continuity[KW85] $\Rightarrow$ Modulus of continuity

X, Y: compact metric spaces with entropies η, θ
Under the criterion: **linearly admissible representations**

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ L\text{-admissible}\uparrow & & L\text{-admissible}\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{F} & \{0,1\}^{\mathbb{N}} \end{array}$$

Main Theorem of Computational Complexity for Continuous Data

f has modulus $\mu \Leftrightarrow \exists F$ has modulus $\eta \circ \mu \circ \theta^{-1}$

Compare with [Kreitz, Weirauch, 1985]

Linear Admissibility

Distinguish reasonable encodings from bad ones

ζ reduces to ξ

$$\begin{array}{ccc} \{0,1\}^{\mathbb{N}} & \xrightarrow{\ \xi\ } & X \\ F \Big\uparrow & \overset{\zeta}{\nearrow} & \\ \{0,1\}^{\mathbb{N}} & & \end{array}$$

$\exists F$ such that $\zeta = \xi \circ F$

Definition

A representation ξ is *linearly admissible* iff

- optimal with respect to entropy
- any uniformly continuous representation ζ is optimally reducible to ξ

Compare with [Kreitz, Weirauch, 1985]

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ \mathit{Cauchy}\Big\uparrow & & \mathit{Cauchy}\Big\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{F} & \{0,1\}^{\mathbb{N}} \end{array}$$

Real Numbers $\mathbb{R}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \mathit{admissible}\Big\uparrow & & \mathit{admissible}\Big\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{F} & \{0,1\}^{\mathbb{N}} \end{array}$$

Topological Spaces X, Y

f continuous $\Leftrightarrow$ F continuous

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ L\!-\!\mathit{admissible}\Big\uparrow & & L\!-\!\mathit{admissible}\Big\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{F} & \{0,1\}^{\mathbb{N}} \end{array}$$

Compact Metric Spaces X, Y

f has modulus $\mu \Leftrightarrow F$ has modulus $\eta \circ \mu \circ \theta^{-1}$

Our Theory is NOT Empty

We have been saying: under the **criterion** of admissibility, good things happen.
What if it is impossible to meet the **criterion**?

Theorem

Every compact metric space admits a linearly admissible representation.

The **criterion** can always be met, good things always happen! We are happy!

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Nonemptiness

Theorem

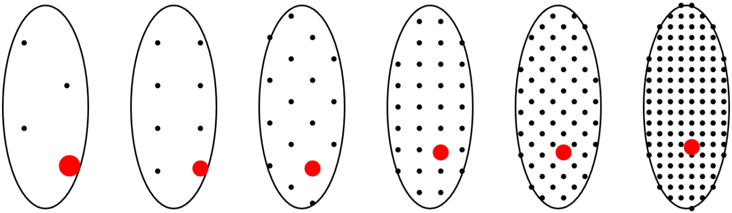
Every compact metric space admits a linearly admissible representation.

Definition

A representation ξ is *linearly admissible* iff

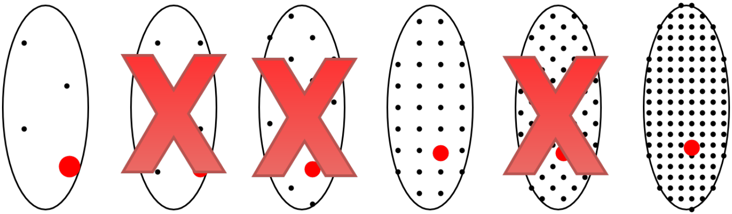
- optimal with respect to entropy
- any uniformly continuous representation ζ is optimally reducible to ξ

Standard Representation



- Encode $x \in X$ as a sequence of points converging to x
- For each precision 2^{-n}, there are $2^{\eta(n)}$ points.
- $\eta(n)$ bits needed to identify one point among $2^{\eta(n)}$
- Concatenate names of points
- modulus of continuity $= \sum_{i \leq n} \eta(i)$

Concise Standard Representation



- modulus $= \sum_{i \leq k} \eta(\varphi(i))$ for appropriate nondecreasing unbounded $\varphi : \mathbb{N} \to \mathbb{N}$
- $\eta(\varphi(k))$ grows like exponential $\Rightarrow \sum_{i \leq k} \eta(\varphi(i))$ dominated by last term
- $\mu(n) = \sum_{i \leq k} \eta(\varphi(i)) \leq C \cdot \eta(n)$
- Careful choice of φ needed
- Arbitrarily fast-growing φ does not work

Admissibility

Theorem

Concise standard representation is linearly admissible.

Proof. By careful analysis and construction.

Main Theorem

Theorem

Let X be a compact metric space with entropy η_X, representation ξ_X, with modulus of continuity μ_X. Similarly for Y, η_Y, ξ_Y, and μ_Y. For $f : X \to Y$,

- f has modulus $\mu_f \Rightarrow \exists$ a realizer F with modulus $\mu_F = \mu_X \circ \mu_f \circ \mu_Y^{-1}$
- $\exists$ a realizer F with modulus $\mu_F \Rightarrow$ f has modulus $\mu_f = \mu_X^{-1} \circ \mu_F \circ \mu_Y$

Proof. Make use of concise standard representation.

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Summary

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle admissible}\uparrow & & {\scriptstyle admissible}\uparrow \\ \{0,1\}^{\mathbb{N}} & \xrightarrow{\;F\;} & \{0,1\}^{\mathbb{N}} \end{array}$$

- $f : [0,1] \to [0,1]$ is oracle computable $\Leftrightarrow$ f continuous $\Leftrightarrow$ F continuous
- Generalization to topological spaces [Kreitz and Weihrauch, 1985]: f continuous $\Leftrightarrow$ F continuous
- Refinement to compact metric spaces [Our Contribution]: f has modulus $\mu \Leftrightarrow F$ has modulus $\eta \circ \mu \circ \theta^{-1}$
- Our theory is not empty
- Proof

Thank you very much for your attention!