Randomized computation in computable analysis

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Random Variable

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If there is no special explanation, Ω is always Cantor space $\mathcal{C}$ with canonical Borel probability measure vol.

Computability of random variable

Definition

Let X be topological space with representation $\xi :\subseteq \mathcal{C} \to X$. A random variable $R :\subseteq \mathcal{C} \to X$ is almost surely computable if there exists $U \subseteq \mathcal{C}$ s.t. $\mathrm{vol}(U) = 1$ and $R|_U : U \to X$ is total computable function with respect to representation ξ.

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Definition

Let $U \subseteq \mathcal{C}$ s.t. $\mathrm{vol}(U) = 1$. A random variable $R : U \to X$ is probabilistically computable if there exists computable sequence $R_n : U \to X$ s.t.

$$\forall n \in IN \;\; \mathrm{vol}\{\omega : d(X(\omega), X_n(\omega)) > 2^{-n}\} < 2^{-n}$$

Convergence of random variables

Definition

Let X be the metric space equipped with metric d. The sequence of random variable $\{R_n\} :\subseteq \mathcal{C} \to X$ converges to $R :\subseteq \mathcal{C} \to X$ *almost surely* if $\text{vol}(\{\omega : d(R_n(\omega), R(\omega)) \to 0\}) = 1$

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Definition

The sequence of random variable $\{R_n\} :\subseteq \mathcal{C} \to X$ converges to $R :\subseteq \mathcal{C} \to X$ *uniformly almost surely* if there exists $U \subseteq \bigcap_n dom(R_n) \cap dom(R)$ s.t. $\mu(U) = 1$ and $\sup_{\omega \in U} d(R_n(\omega), R(\omega)) \to 0$.

Almost sure Computability of random variable

Lemma

Suppose $U \subseteq \mathcal{C}$ has measure 1 and $R_n : U \to X$ is a computable sequence with respect to representation ξ and there exists recursive $\nu : \mathbb{N} \to \mathbb{N}$ s.t.

$$\forall n \geq \nu(m) : \sup_{\omega \in U} d\big(R_n(\omega), R(\omega)\big) \leq 2^{-m}$$

Then R is almost surely computable

Probabilistic Name

Definition (Schröder,Simpson 2005)

A probabilistic name π on $\mathcal{C}$ is Borel probability measure on $\mathcal{C}$ such that

$$\pi(\mathcal{C}) = 1 \text{ and } \pi(w\mathcal{C}) = \pi(w0\mathcal{C}) + \pi(w1\mathcal{C}) \text{ for all } w \in \{0,1\}^*$$

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Theorem (Schröder,Simpson 2005)

Let (X, u) be second countable space with admissible representation ξ and Borel probability measure μ. Then there exists probabilistic name π on $dom(\xi)$ s.t. $\mu(S) = \pi(\xi^{-1}(S))$ for every measurable subset of X.

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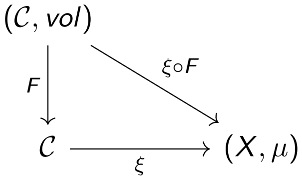
Can we use vol, instead of π?

Realizer of probability measure

Definition

Let $\mathcal{C} := \{0,1\}^\omega$ denote Cantor space, equipped with the canonical Borel probability measure vol. Let X denote a topological space with representation $\xi :\subseteq \mathcal{C} \twoheadrightarrow X$ and Borel probability measure μ. We say that $F :\subseteq \mathcal{C} \to \mathit{dom}(\xi)$ is a *ξ-realizer* of μ if $\mu(S) = \mathrm{vol}\left(F^{-1}\left[\xi^{-1}[S]\right]\right)$ holds for every Borel $S \subseteq X$. Call μ *computable* if it has a computable realizer.



Example

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$\rho : \bar{b} \mapsto \sum_{j \geq 0} b_j 2^{-j-1}$ is representation of unit interval $[0;1]$.
Then, identity function $I : \mathcal{C} \to \mathcal{C}$ is computable ρ-realizer of the Lebesgue measure on $[0;1]$.

Our conjecture

Conjecture

Let representation ξ of X be admissible for second-countable Hausdorff space X and μ a Borel probability measure. Then there exists a realizer F of μ w.r.t. ξ.

Stochastic Process

Definition

Random variable is function $X : \Omega \to \mathbb{R}$ where Ω is probability space.

Definition (Stochastic process)

Stochastic process is function $S : \Omega \to X$ where Ω is probability space and X is space of functions.

Wiener process as random variable

Definition

Wiener process W is a random variable $W : \mathcal{C} \to \mathcal{C}_0[0;1]$ where $\mathcal{C}_0[0;1]$ is space of continuous functions $f : [0;1] \to \mathbb{R}$ s.t. $f(0) = 0$ with uniform metric.

Schauder Function

Definition (Schauder function)

Schauder Function $\{F_0, F_{n,j}\}$ is defined by

$$F_0(t) = t$$

and

$$F_{n,j}(t) = \begin{cases} 2^{(n-1)/2}(t - \frac{k-1}{2^n}) & \frac{k-1}{2^n} \leq t \leq \frac{k}{2^n} \\ 2^{(n-1)/2}(\frac{k+1}{2^n} - t) & \frac{k}{2^n} \leq t \leq \frac{k+1}{2^n} \\ 0 & otherwise \end{cases}$$

where $k = 2j - 1$

Levy's construction

Fact

Let R_0 and $R_{n,j}$ be independent gaussian random variables. Following sequence converges to the Wiener process almost surely:

$$W_N(t) = R_0 t + \sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} R_{n,j} \varphi_{n,j}(t)$$

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Lemma

The sequence W_N does not converge to Wiener Process W uniformly.

Almost sure Computability of random variable

Lemma

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$$\forall n \geq \nu(m) : \sup_{\omega \in U} d\big(R_n(\omega), R(\omega)\big) \leq 2^{-m}$$

Then R *is almost surely computable*

Probabilistic computability of Wiener process

Lemma

Let $W_N(t) = R_0 t + \sum_{n=1}^{N} \sum_{j=1}^{2^{n-1}} R_{n,j}\varphi_{n,j}(t)$ *where* R_0 *and* $R_{n,j}$ *is independent gaussian random variables.*

$$P\left(||W - W_N||_\infty \le \frac{\sqrt{2\log 2}}{(1-2^{-1/4})\cdot 2^{m/4}} \right) \ge 1 - \frac{2a\cdot 2^{-m}}{\sqrt{m}}$$

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Theorem

Wiener process W *is probabilistically computable.*

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- Probabilistic name and realizer

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- Probabilistic name and realizer
- Wiener process and its computability

Thank you!