임동현, Martin Ziegler

Quantitative Coding and Complexity Theory of Continuous Data

with thanks to:
Akitoshi Kawamura
Matthias Schröder
Svetlana Selivanova
Florian Steinberg

Computer Science

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Implementation/library

Abstract Data Types

Semantics/Verification
(Scott, Hoare, …)

Complexity
(Hartmanis, Cook, …)

Computability
(Turing, Kleene, …)

Specification

discrete (integers, graphs, …)

real (numbers, functions, operators)

continuous structures

„A good ***definition*** *is worth 1000* ***lemmas***"
(Doron Zeilberger, Opinion #82)

Encode graph $G=(V,E)$ as: ▪ adjacency matrix ▪ list of edges (pairs of vertices) ▪ rotation map:

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all mutually polynomial-time (logspace) equivalent

Encode integer in ▪ binary vs. ▪ unary:
computably equivalent, highly different *complexity*

[Folklore] Any computable $f:[0;1]\to[0;1]$ is continuous.

[Ko'91, §2.4] If $f:[0;1]\to[0;1]$ is computable in *polynom.* time, then it has a *polynomial* modulus of continuity.

[Pour-El&Richards'89, Weihrauch&Zhong, …]
Qualitative encoding/computability on Banach spaces.

[Steinberg'17], [Lim,Selivanova,Z'20]: What is a *polynomial-time* computable Sobolev / $\mathcal{L}^2$ function?

Encoding Reals [0;1]

binary: $r = \sum_n c_n 2^{-n}, \ \hat{c}=(c_n) \in C = \{0,1\}^{\mathbb{N}}$

Addition *un*computable [Tur'37]

not (at all) unique

rational: $|r - a_{2n}/a_{2n+1}| \leq 2^{-n}, \ a_n \in \mathbb{N}$

Addition computable, but *not in bounded* time [Wei00,§7.2]

$(\mathrm{bin}(a_n)) \in C$

dyadic: $|r - a_n/2^n| \leq 2^{-n}, \ a_n \in \mathbb{N}$

To access/produce 2^{-n} approx., skip over $O(n^2)$ bits

signed: $r = \sum_n s_n \cdot 2^{-n}, \hat{s}=(s_n) \in \{\mathbf{-1},0,+1\}^{\omega} \subseteq C$

Addition is computable in *linear* time (by a DFA)

Overview, Part I

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- Encoding reals: binary, rational, dyadic, signed
- Qua**nt**itative (=*skolemized* qualitative) continuity
- Computing on "basic" spaces: Cantor, Baire, …
- Abstract coding: representations and realizers
- Qua**l**itative ***admissibility*** and
  qua**l**itative ***Main Theorem*** [Kreitz-Weihrauch'85]
- qua**nt**itative *Main Theorem* over the reals [Ko91]
- Goal: qua**nt**itative admissibility and *Main Theorem* for generic compact metric spaces

Quantitative Continuity

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Def [Folklore]: Fix metric spaces (X,d) and (Y,e).

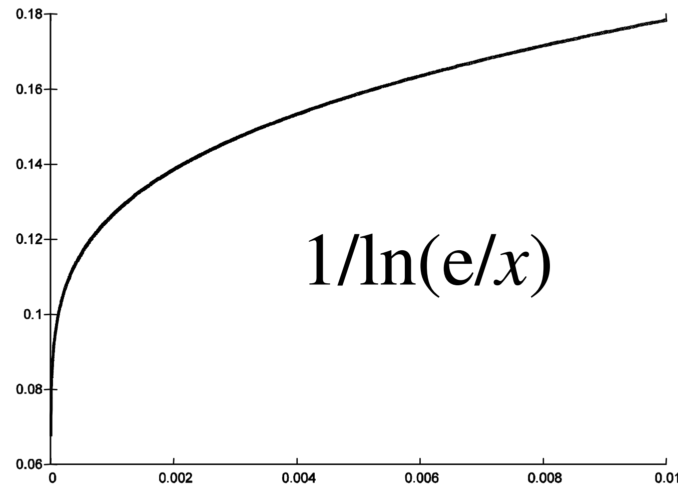
A modulus of continuity of $f:(X,d)\to(Y,e)$ is any $\mu:\mathbb{N}\to\mathbb{N}$ such that $\forall n\geq n_0$ $d(x,x')\leq 2^{-\mu(n)}$ implies $e(f(x),f(x'))\leq 2^{-n}$

If $f:X\to Y$ has μ and $g:Y\to Z$ has ν, then $g\circ f$ has $\mu\circ\nu$.

Example: Lipschitz-continuous $\Leftrightarrow$ modulus $\mu(n)\leq n+O(1)$

b) Hölder-continuous $\Leftrightarrow$
modulus $\mu(n)\leq O(n)$

c) h: $[0;1]\ni x\to 1/\ln(e/x)\in[0;1]$
has (only) exponential modulus.



d) $h\circ h$: (only) *doubly* exponential modulus.

Computing on "*Basic*" Spaces

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Def (rel.cost): Call F computable in ***relative*** *polyn.time* if F is computable in time $\mathrm{poly}(n+\boldsymbol{\mu})$ for F's least modulus μ

Type-2 Machine computes partial $F :\subseteq \mathcal{C}=\{0,1\}^{\{1\}^*} \to \mathcal{C}$

Hyper Machine computes $F :\subseteq \mathcal{H}=\{0,1\}^{\{0,1\}^{\{0,1\}^*}} \to \mathcal{H}$

Oracle Machine computes $F :\subseteq \mathcal{B}=\{0,1\}^{\{0,1\}^*} \to \mathcal{B}$

Computational *cost* metric $d(\varphi,\psi) = 2^{-\min\{\, |\underline{x}| \,:\, \varphi(\underline{x})\neq\psi(\underline{x}) \,\}}$

3 *Main Properties* of computing on "basic" spaces $\mathcal{U}$:

Computing on compact $\mathrm{dom}(F)$ has a time bound $t:\mathbb{N}\to\mathbb{N}$.

Any F computable in time $t:\mathbb{N}\to\mathbb{N}$ has modulus $\leq t(n+O(1))$

"*Main Thm*": Any computable $F:\subseteq\mathcal{U}\to\mathcal{U}$ is continuous.

"Basic" Function Spaces

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$$C^\dagger := \mathrm{Lip}_1(C,C) = \{\ F\!:\!C\to C \mid d(F(x),F(x')) \le d(x,x')\ \}$$

$$\mathrm{Appl} : C^\dagger \times C \ni (F,\underline{x}) \to F(\underline{x}) \in C$$

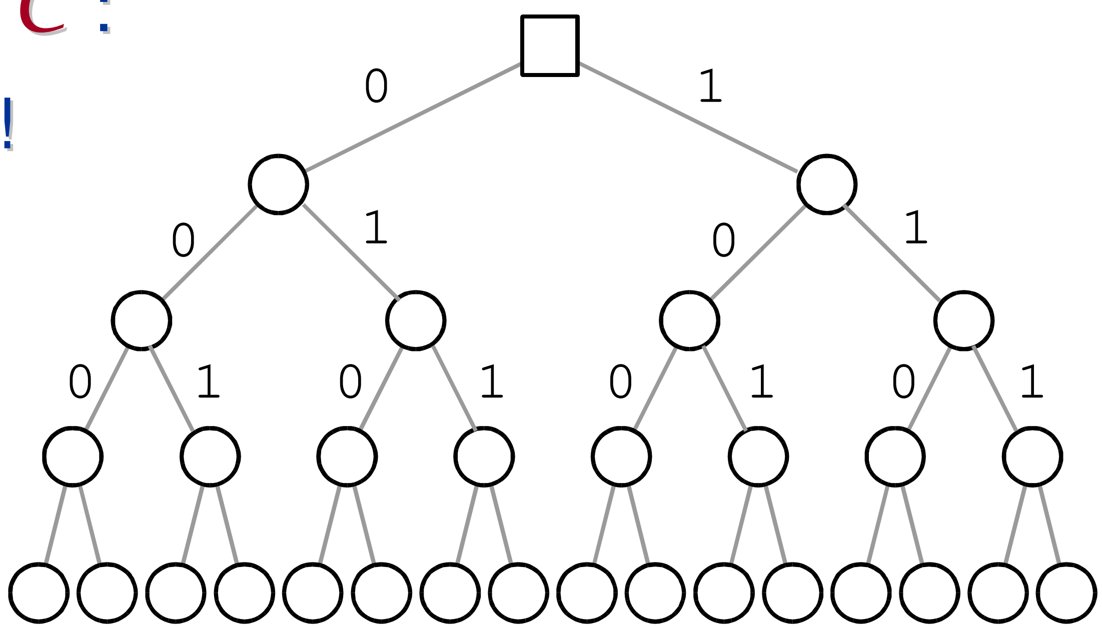
- F "*is*" a complete binary tree with nodes labeled $\{\mathbf{0},\mathbf{1}\}$

  F "$\in$" $\mathcal{B}$, Appl: $\mathcal{B}\times C\to C$ polyn.time oracle computable

- "*enumerating*" F "$\in$" C :

  Appl **not** polynom.time!

- $C=\{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$
- $\mathcal{B}=\{\mathbf{0},\mathbf{1}\}^{\{0,1\}^*}=C^\dagger$
- $\mathcal{H}=\mathcal{B}^\dagger$



Representations & Realizers

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Def: [Kreitz&Weihrauch'85] A **representation** of a space X is a partial surjective mapping $\xi :\subseteq \mathcal{C}=\{0,1\}^{\mathbb{N}} \to X$.

Type-2 Machine computes partial $F:\subseteq\{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$

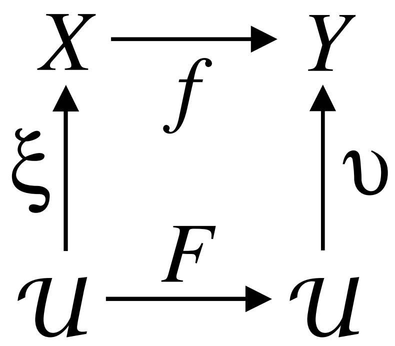
[Kawamura&Cook'12] surjective $\Xi:\subseteq \mathcal{B}:=\{0,1\}^{\{0,1\}^*} \to X$

Oracle Machine computes $F:\subseteq\{0,1\}^{\{0,1\}^*} \to \{0,1\}^{\{0,1\}^*}$

[Bauer,Birkedal,Scott'04] $\xi:\subseteq\mathcal{E}\to X$, $\mathcal{E}$ *equilogical* space

Fix representations ξ of X and υ of Y.

A (ξ,υ)-realizer of $f:X\to Y$ is any *restriction* $F:\mathrm{dom}(\xi)\to\mathrm{dom}(\upsilon)$ s.t. $f\circ\xi \sqsubseteq \upsilon\circ F$.



Computing $f:X\to Y$ means to compute a (ξ,υ)-realizer.

Admissibility, "Main Theorem"

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Def: [Kreitz&Weihrauch'85] Fix representations ξ,ζ of X.
Write $\zeta \preccurlyeq \xi$ iff $\zeta \sqsubseteq \xi \circ F$ for a *continuous* F:$\mathrm{dom}(\zeta) \rightarrow \mathrm{dom}(\xi)$.

Call represent. ξ of X admissible if **(i)** ξ is continuous and **(ii)** every continuous representation ζ of X has $\zeta \preccurlyeq \xi$.

~~**binary:**~~	**not** *admissible*	$\beta :\subseteq C \rightarrow [0;1]$
rational:	*admissible*	$\rho :\subseteq C \rightarrow [0;1]$
dyadic:	*admissible*	$\delta :\subseteq C \rightarrow [0;1]$
signed:	*admissible*	$\sigma :\subseteq C \rightarrow [0;1]$

Main Theorem [KW85] For *admissible* ξ of X and υ of Y, f:$X \rightarrow Y$ is continuous **iff** it has a continuous (ξ,υ)-realizer.

Qualitative → *Quantitative*

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Def: [Kreitz&Weihrauch'85] Fix representations ξ,ζ of X.
Write $\zeta \preccurlyeq \xi$ iff $\zeta \sqsubseteq \xi \circ F$ for a *continuous* F:dom(ζ)→dom(ξ).

Call represent. ξ of X admissible if **(i)** ξ is continuous
and **(ii)** every continuous representation ζ of X has $\zeta \preccurlyeq \xi$.

~~**binary:**~~	**`addition`**: *un*computable	$\beta :\subseteq C \to [0;1]$
rational:	computable, no time bound	$\rho :\subseteq C \to [0;1]$
dyadic:	polynom.time computable	$\delta :\subseteq C \to [0;1]$
signed:	linear-time computable	$\sigma :\subseteq C \to [0;1]$

Real Quantitat. ***Main Thm 2:*** $f:[0;1]\to[0;1]$ has *linear* modulus iff it **has a (σ,σ)-realizer** with *linear* modulus.

Summary, Part I

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- Computing on "basic" spaces $\{0,1\}^{\mathbb{N}}$, $\{0,1\}^{\{0,1\}^*}$,..
  - Single cost parameter $n \in \mathbb{N}$ on compact domains.
  - Quantitative continuity ≈ computational bit-cost:
  - "relative" polynomial-time computability
- Coding other spaces: representations, realizers
  - Qualitative ***admissibility*** and ***Main Theorem***

Real case $[0;1]$		$f:[0;1]\to[0;1]$ vs. realizer F
binary	**not** admissible	continuity ***un***related
rational	*admissible*	qualitatively related continuity
dyadic	*"polynom. admissible"*	polynom. related modulus [Ko]
signdigit	***"linearly admissible"***	**linearly related modulus**

Questions for Part II

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a) What are (criteria for) *complexity*-theoretically "reasonable" encodings of spaces X beyond $[0;1]$?

b) How *quantitatively* refine the qualitative "admissibility" of [Kreitz&Weihrauch'85] ?

c) What is a *quantitative* "Main Theorem" beyond $\mathbb{R}$?

d) How define *polynom.-time* computation beyond $\mathbb{R}$?

e) Which "basic" spaces (Cantor, Baire, …) suitable domains for encoding which spaces X beyond $\mathbb{R}$?

Hierarchy of interest. spaces: $X' = \mathrm{Lip}_1(X,[0;1])$

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Recap of Part I

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- Computing on "basic" spaces $\{0,1\}^{\mathbb{N}}$, $\{0,1\}^{\{0,1\}^*}$,..
  - single cost parameter $n\in\mathbb{N}$ on <u>compact</u> domains.
  - computational cost $t(n)$ → modulus of continuity μ
  - "relative" polynomial-time: $t(n) \leq \text{poly}(n+\mu(\text{poly } n))$
- Coding other spaces: representations, realizers
  - qualitative ***admissibility*** and ***Main Theorem***

Real case $[0;1]$		$f:[0;1]\to[0;1]$ vs. realizer F
binary	**not** admissible	continuity ***un***related
rational	*admissible*	qualitatively related continuity
dyadic	*"polynom. admissible"*	polynom. related modulus [Ko]
signdigit	***"linearly admissible"***	**linearly related modulus**

Overview, Part II

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- Recap: qua**l**itative admissibility, qua**nt**itative real *Main Theorem*, *relative* polynom.-time

- *Polynomial* admissibility*:*
  - (relative) polynomial reduction $\zeta \preccurlyeq_{\mathbf{p}} \xi$
  - tentative polynomial *Main Theorem*
  - deficiencies

qua**l**itative	qua**nt**itative
computability	complexity
topology	metric
(uniformly) continuous	modulus of continuity
compact	entropy
continuous image of compact	Steinberg's Lemma

Compactness & Continuity

Def (tentative): Call ξ *polynomially* admissible if
(i) ξ has compact domain and polyn. modulus $\mu(n)\leq$ ~~$\text{poly}(n)$~~

Call represent. ξ of X admissible if **(i)** ξ is continuous

Main Properties of computing on "basic" spaces $\mathcal{U}$:

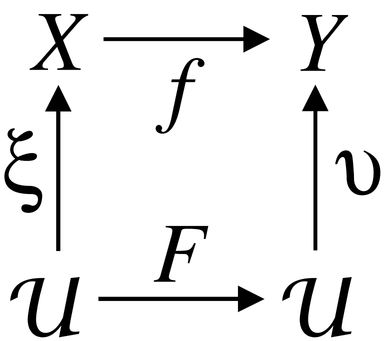
Computing on compact $\text{dom}(f)=X$ has time bound $t:\mathbb{N}\to\mathbb{N}$.

Any $f:X\to Y$ computable in time $t:\mathbb{N}\to\mathbb{N}$ has modulus **???**

Fix representations ξ of X and υ of Y.

A (ξ,υ)-realizer of $f:X\to Y$ is any *restriction*

$F:\text{dom}(\xi)\to\text{dom}(\upsilon)$ s.t. $f\circ\xi \sqsubseteq \upsilon\circ F$.



Computing $f:X\to Y$ means to compute a (ξ,υ)-realizer.

Entropy = Quantitative Compactness

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Def (tentative): Call ξ *polynomially* admissible if
(i) ξ has compact domain and modulus $\mu(n)\leq\text{poly}(\eta)$

Def [Kolmogorov'59]: Entropy of (X,d) is $\eta:\mathbb{N}\to\mathbb{N}$ s.t.
X can be covered by $2^{\eta(n)}$, but no less, balls of radius 2^{-n}.

Lemma [Steinberg'17]: If $f:Y\to Z$ has modulus μ and if Y has entropy η, then $\text{range}(f)\subseteq Z$ has entropy $\leq\eta\circ\mu$.

Any representation $\delta:\subseteq C\to[0;1]'$ *must* have exp. modulus!
"*Therefore*" [Kawamura/Cook'12] consider $\delta_\square:\subseteq\mathcal{B}\to[0;1]'$

Examples: a) $[-2^k;2^k]^d$ has entropy $\eta(n)=\Theta(d\cdot(n+k))$

b) $C=\{0,1\}^{\mathbb{N}}$ has entropy $\eta(n)=n$. **d)** $\mathcal{H}$ has $\eta(n)=2^{2^n}$.

e) $X^\dagger=\text{Lip}_1(X,C)$ and $X'=\text{Lip}_1(X,[0;1])$ have entropy
$\eta^\dagger(n)=2^{\text{poly}(n+\eta(\Theta(n)))}$. **c)** $\mathcal{B}=\{0,1\}^{\{0,1\}^*}$ has $\eta(n)=2^n$.

Tentative Polynomial Admissibility

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Def (tentative): Call ξ *polynomially* admissible if
(i) ξ has compact domain, modulus $\mu(n) \leq \mathrm{poly}(n+\eta(\mathrm{poly}(n)))$
(ii) and every representation ζ of X has $\zeta \preccurlyeq_{\mathbf{p}} \xi$

For representat.s ζ,ξ of X with <u>least</u> moduli ν,μ write $\zeta \preccurlyeq_{\mathbf{p}} \xi$
if there exists $F:\mathrm{dom}(\zeta)\to\mathrm{dom}(\xi)$ with modulus κ s.t. $\zeta \sqsubseteq \xi \circ F$
and ~~$\kappa(\mu(n)) \leq \nu(n)$.~~

$\Rightarrow \nu \leq \kappa \circ \mu$

and $\kappa(\mu(n)) \leq \nu(\mathrm{poly}(n))$.
and ~~$\kappa(\mu(n)) \leq \mathrm{poly}(n+\nu(\mathrm{poly}(n)))$.~~

Call represent. ξ of X admissible if **(i)** ξ is continuous
and **(ii)** every continuous representation ζ of X has $\zeta \preccurlyeq \xi$.

Write $\zeta \preccurlyeq \xi$ iff $\zeta \sqsubseteq \xi \circ F$ for a *continuous* $F:\mathrm{dom}(\zeta)\to\mathrm{dom}(\xi)$.

If $f:X\to Y$ has μ and $g:Y\to Z$ has ν, then $g\circ f$ has $\mu\circ\nu$.

Tentative Polynomial Main Theorem

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Def (tentative): Call ξ <u>*polynomially*</u> admissible if

(i) ξ has compact domain, modulus $\mu(n)\leq\mathrm{poly}(n+\eta(\mathrm{poly}(n)))$

(ii) and every representation ζ of X has $\zeta \preccurlyeq_{\mathbf{p}} \xi$

Theorem [Kawamura, Lim, Steinberg, Z. 2016~2019]

a) Every compact (X,d) <u>has</u> a polynomially admissible representation ξ.

$n\to\min\{m\colon \nu(m)\geq n\}$

b) "$\preccurlyeq_{\mathbf{p}}$" is transitive.

c) Let ξ,υ be polynom.admissible for X and Y, respectively with <u>least</u> moduli μ,ν. If $f{:}X\to Y$ has modulus κ, then it has a (ξ,υ)-realizer F with modulus $\lambda \leq \mu\circ\kappa\circ\mathrm{poly}\circ\nu^{-1}$ and

d) If F has modulus λ, then f has modulus $\kappa \leq \mathrm{poly}\circ\mu^{-1}\circ\lambda\circ\nu$.

Main Theorem [KW85] For *admissible* ξ of X and υ of Y, $f{:}X\to Y$ is continuous **iff** it has a continuous (ξ,υ)-realizer.

Deficiencies

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Def (tentative): Call ξ *polynomially* admissible if

(i) ξ has compact domain, modulus $\mu(n) \leq O(\eta(n+O(1)))$

(ii) and ~~every representation ζ of X has $\zeta \preccurlyeq_{\mathbf{p}} \xi$~~ $\xi^{-1} \circ \zeta \sqsubseteq F$ selection

Theorem [Kawamura/Lim/Steinberg/Z. 2016~19]

a) Every compact (X,d) has a polynomially admissible representation ξ.

extensional: only (inverse) entropies

b) "$\preccurlyeq_{\mathbf{p}}$" is transitive.

c) Let ξ,υ be polynom.admissible for X and Y, respectively with least moduli μ,ν. If $f{:}X \to Y$ has modulus κ, then it has a (ξ,υ)-realizer F with modulus $\lambda \leq \mu \circ \kappa \circ \text{poly} \circ \nu^{-1}$ and

d) If F has modulus λ, then f has $\kappa \leq \text{poly} \circ \mu^{-1} \circ \mu \circ \kappa \circ \text{poly} \circ \nu^{-1} \circ \nu$

- *In*tensional bounds: (inverse) moduli of representations.
- How improve *polynomial* to *linear*? "Loss" (c)$\Rightarrow$(d)$\Rightarrow$(c)

Recap of Part II

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- Recap: qua**l**itative admissibility, qua**nt**itative real *Main Theorem*, *relative* polynom.-time

- *Polynomial* admissibility*:*
  - (relative) polynomial reduction $\zeta \preccurlyeq_p \xi$
  - tentative polynomial *Main Theorem*
  - deficiencies

qua**l**itative	qua**nt**itative
computability	complexity
topology	metric
(uniformly) continuous	modulus of continuity
compact	entropy
continuous image of compact	Steinberg's Lemma
equilogical space [BBS]	compact ultrametric

Overview, Part III

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- Recap: polynomial admissibility, (intensional) polynomial *Main Theorem*
- *Multi*functions
  - and their (quantitative) continuity
- **Quantitative continuous selection theorem** for multifunctions between compact *ultra*metric spaces

  = domains of *generalized* representations
- "*In*tensional" linear *Main Theorem*
- "*Ex*tensional" polynomial *Main Theorem* ?

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Overview, Part III

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- Recap of Part II: polynomial admissibility, *in*tensional <u>quantitative</u> *Main Theorem*
- Goal: *ex*tensional (tight?) Main Theorem
- *Multi*functions
  - and their (quantitative) continuity
- **Quantitative continuous selection theorem** for multifunctions between compact *ultra*metric spaces

  = domains of *generalized* representations
- "*In*tensional" <u>linear</u> *Main Theorem*

Recap of Part II

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***In**tensional* **Polynomial** ***Main Theorem*:**

Let X and Y compact metric spaces with *polynomially* admissible representations ξ of X and υ of Y, least moduli of continuity μ,ν of ξ,υ; $f:X\to Y$

a) If f has a κ-continuous realizer, it is $\mu\circ\kappa\circ\text{poly}\circ\nu^{-1}$-continuous.

b) λ-continuous f has a realizer with modulus $\text{poly}\circ\mu^{-1}\circ\lambda\circ\nu$.

"*Loss*" $\kappa\leq\text{poly}\circ\mu^{-1}\circ\mu\circ\kappa\circ\text{poly}\circ\nu^{-1}\circ\nu$

qualitative	quantitative
computability	complexity
topology	metric
(uniformly) continuous	modulus of continuity
compact	entropy
continuous image of compact	Steinberg's Lemma
equilogical space [BBS]	compact ultrametric

Goal: *Ext*ension. Main Thm

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Hope for ***Ext*ensional Quantitative *Main Theorem*:**

$\xi^{-1}\circ\zeta \sqsubseteq F$
selection

Let X and Y compact metric spaces with entropies η and θ.

There *exist* representations ξ of X and υ of Y such that:

such that $\Lambda(\eta,\theta,K(\eta,\theta,\lambda)) \approx \lambda$
and $K(\eta,\theta,\Lambda(\eta,\theta,\kappa)) \approx \kappa$

a) If f has a κ-continuous realizer,
it is $\Lambda(\eta,\theta,\kappa)$-continuous

b) λ-continuous f has a realizer
with modulus $K(\eta,\theta,\lambda)$

For representat.s ζ,ξ of X with <u>least</u> moduli ν,μ write $\zeta \preccurlyeq_p \xi$
if there exists $F:\mathrm{dom}(\zeta)\to\mathrm{dom}(\xi)$ with modulus κ s.t. $\zeta \sqsubseteq \xi\circ F$

Multifunctions

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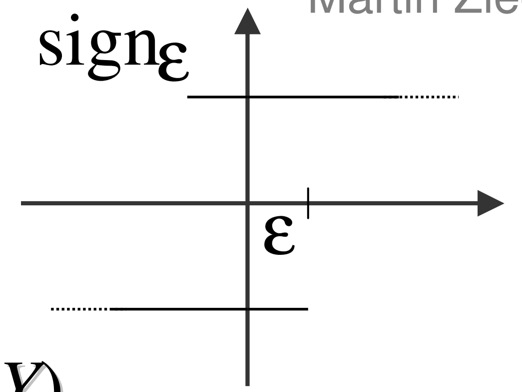
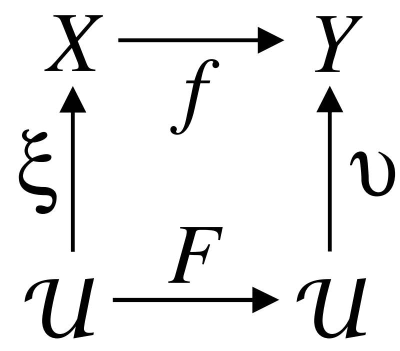
- Aka non-extensional "functions"
- Unavoidable in real computation
- Partial *multi*function $G :\subseteq X \rightrightarrows Y$ is a relation $G \subseteq X \times Y$ or *set*function $G : X \to \mathcal{P}(Y)$



sign_0 has *no* continuous realizer!



Function problem G: ;

Input x, output $y = G(x)$;

$\upsilon^{-1} \circ f \circ \xi \sqsubseteq F$

$f \sqsubseteq \upsilon \circ F \circ \xi^{-1}$

$\xi^{-1} \circ \zeta \sqsubseteq F$

(ξ, υ)-realizer of $f : X \rightrightarrows Y$ is $F : \text{dom}(\xi) \to \text{dom}(\upsilon)$ st. $f \circ \xi \sqsubseteq \upsilon \circ F$

[KW85] $\zeta \preccurlyeq \xi$ iff $\zeta \sqsubseteq \xi \circ F$ for *continuous* $F : \text{dom}(\zeta) \to \text{dom}(\xi)$.

Continuity of Multifunctions

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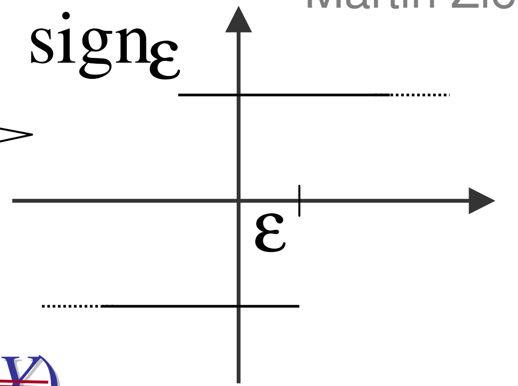
Vietoris pseudo-quasi-metric: see below

*dis*contin. w.r.t Hausdorff metric

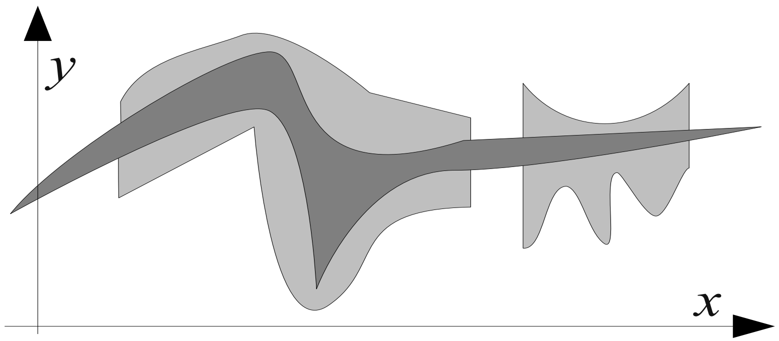


- Partial *multi*function $G{:}\subseteq X \rightrightarrows Y$ is a relation $G \subseteq X \times Y$ or ~~setfunction $G{:}X \to \mathcal{P}(Y)$~~

- Restriction $F \sqsubseteq G$: smaller domain and/or *larger* range(s).

"easier"

Function problem:
Input x, output $y = G(x)$;
not necessarily *all* $y \in G(x)$



Multi-Function G ~~*he*~~*mi-* continuous
$\Rightarrow$ so is $F \sqsubseteq G$.

Call G *compact* if its graph is.

Quantitative Continuity *of Multifunctions*

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$\exists n_0 \in \mathbb{N}\ \ \forall x_0 \in X\ \ \exists y_0 \in G(x_0)$ for every natural number k

$\forall n_1 \geq n_0\ \ \forall x_1 \in \underline{B}_{\mu(n_1)}(x_0)\ \ \exists y_1 \in G(x_1) \cap \underline{B}_{n_1}(y_0)$

$\forall n_2 \geq n_1 + n_0\ \ \forall x_2 \in \underline{B}_{\mu(n_2)}(x_1)\ \ \exists y_2 \in G(x_2) \cap \underline{B}_{n_2}(y_1)$

$\cdots\ \ \forall n_k \geq n_{k\text{-}1} + n_0\ \ \forall x_k \in \underline{B}_{\mu(n_k)}(x_{k\text{-}1})\ \ \exists y_k \in G(x_k) \cap \underline{B}_{n_k}(y_{k\text{-}1})$

closed under

- Restriction $F \sqsubseteq G$: smaller domain and/or *larger* range(s).

generalizes *single*-valued

"easier"

Search problem G: Input x, output *some/any* $y \in G(x)$;

$\underline{B}_m(x) :=$
$\{\ x' : d(x,x') \leq 1/2^m\ \}$

- Partial *multi*function $G :\subseteq X \rightrightarrows Y$

μ:ℕ→ℕ strictly increasing:
closed under composition

A modulus of continuity of $G:(X,d) \to (Y,e)$ is $\mu:\mathbb{N}\to\mathbb{N}$ s.t.

$n_0 \mid n \Rightarrow \quad \forall x' \in \underline{B}_{\mu(n)}(x) \quad \Rightarrow \quad y' \in \underline{B}_n(y) \quad , \ y = G(x),\ y' = G(x')$

Selection in a Multifunction

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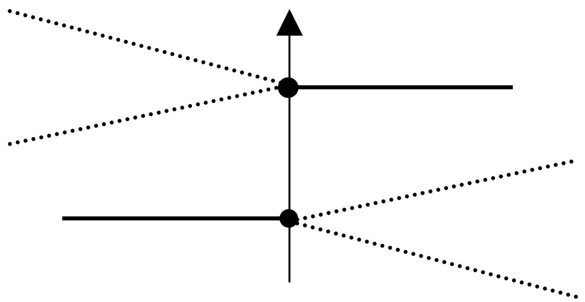
Total $G{:}X\rightrightarrows Y$

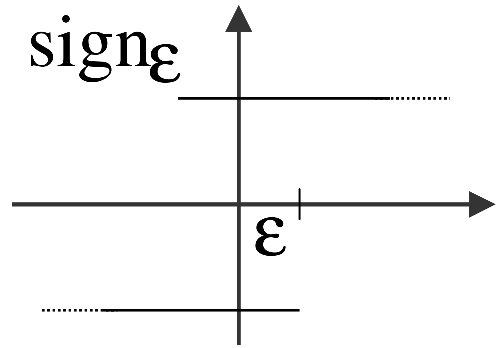
$\exists n_0\in\mathbb{N}\ \ \forall x_0\in X\ \ \exists y_0\in G(x_0)$

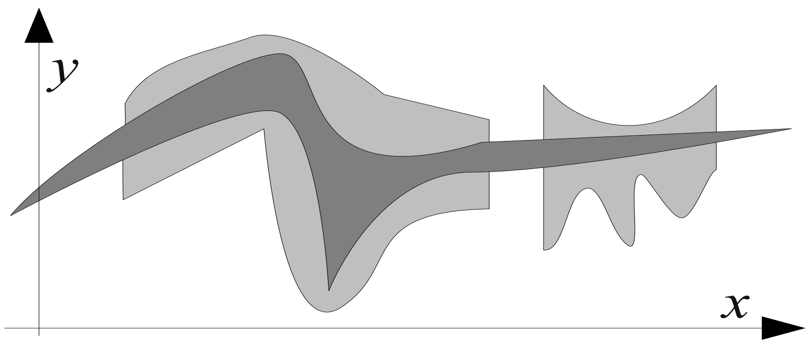
$\forall n_1\geq n_0\ \ \forall x_1\in \underline{B}_{\mu(n_1)}(x_0)\ \ \exists y_1\in G(x_1)\cap \underline{B}_{n_1}(y_0)$

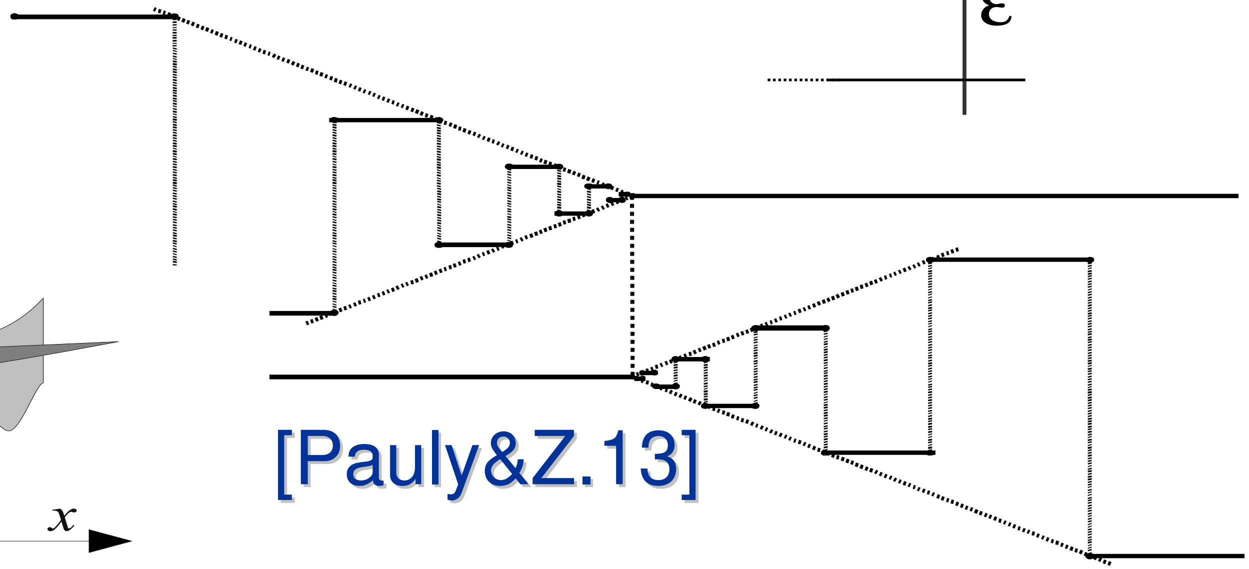
$\forall n_2\geq n_1+n_0\ \ \forall x_2\in \underline{B}_{\mu(n_2)}(x_1)\ \ \exists y_2\in G(x_2)\cap \underline{B}_{n_2}(y_1)$



Selection of $G{:}X\rightrightarrows Y$
total $g{:}X\to Y$ s.t. $G\sqsubseteq g$







[Pauly&Z.13]

[KW85] $\zeta\preccurlyeq\xi$ iff $\xi^{-1}\circ\zeta\sqsubseteq F$ for *continuous* $F{:}\mathrm{dom}(\zeta)\to\mathrm{dom}(\xi)$

(ξ,υ)-realizer of $f{:}X\rightrightarrows Y$ is $F{:}\mathrm{dom}(\xi)\to\mathrm{dom}(\upsilon)$ st. $f\circ\xi\sqsubseteq\upsilon\circ F$

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$$\exists n_0\in\mathbb{N}\ \ \forall x_0\in X\ \ \exists y_0\in G(x_0) \qquad \text{for every natural number } k$$

$$\forall n_1\geq n_0\ \ \forall x_1\in \underline{B}_{\boldsymbol{\mu}(n_1)}(x_0)\ \ \exists y_1\in G(x_1)\cap \underline{B}_{n_1}(y_0) \qquad \textbf{(1)}$$

$$\forall n_2\geq n_1+n_0\ \ \forall x_2\in \underline{B}_{\boldsymbol{\mu}(n_2)}(x_1)\ \ \exists y_2\in G(x_2)\cap \underline{B}_{n_2}(y_1)$$

$$\cdots\quad \forall n_k\geq n_{k-1}+n_0\ \ \forall x_k\in \underline{B}_{\boldsymbol{\mu}(n_k)}(x_{k-1})\ \ \exists y_k\in G(x_k)\cap \underline{B}_{n_k}(y_{k-1})$$

SELECTION THEOREM [Lim&Z., `arXiv:2002.04005`]:
Let $(\mathcal{D},d)$ and $(\mathcal{E},e)$ denote compact *ultra*metric spaces.
If total compact $G{:}\mathcal{D}\rightrightarrows\mathcal{E}$ satisfies (1) for strictly increasing μ,
it admits a selection g with modulus of continuity $\mu(n+O(1))$.

***In*tensional Quant. *Main Thm*:** Let X and Y compact metric spaces, total representations $\xi{:}\mathcal{D}\to X$ and $\upsilon{:}\mathcal{E}\to Y$, and strictly increasing moduli of continuity μ,ν of ξ,υ and μ',ν' of ξ^{-1},υ^{-1}.

a) If $f{:}X\rightrightarrows Y$ has a κ-continuous realizer, it is $\mu'\circ\kappa\circ\nu$-continuous.

b) Compact λ-continuous f has $\mu\circ\lambda\circ\nu'(n+O(1))$-contin. realizer.

Example: Linear Real *Main Theorem*

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Polynomial Real *Main Theorem* [Ko91]:

$f:[0;1]\to[0;1]$ has modulus $\mathrm{poly}(\lambda(\mathrm{poly}(n)))$ iff it has a (δ,δ)-realizer with modulus $\mathrm{poly}(\lambda(\mathrm{poly}(n)))$

- Cantor space C is *ultra*metric compact.
- Signed digit $\sigma:\subseteq C\to[0;1]$ has a linear modulus of continuity;
- and so does its multivalued inverse $\sigma^{-1}:[0;1]\to C$. [nontrivial!]

compact domain and

Linear Real *Main Theorem*: $f:[0;1]\rightrightarrows[0;1]$ has modulus $O(\lambda(O(n)))$ iff it has a (σ,σ)-realizer with modulus $O(\lambda(O(n)))$.

***In*tensional Linear *Main Thm*:** Let X and Y compact metric spaces, total representations $\xi:\mathcal{D}\to X$ and $\upsilon:\mathcal{E}\to Y$, and strictly increasing moduli of continuity μ,ν of ξ,υ and μ',ν' of ξ^{-1},υ^{-1}.

a) If $f:X\rightrightarrows Y$ has a κ-continuous realizer, it is $\mu'\circ\kappa\circ\nu$-continuous.

b) Compact λ-continuous f has $\mu\circ\lambda\circ\nu'(n+O(1))$-contin. realizer.

Goal / Vision

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Hope for ***Ex*tensional Quantitative *Main Theorem*:**
Let X and Y be compact metric spaces with entropy η and θ.
There exist representations ξ of X and υ of Y such that:
a) If $f{:}X\to Y$ has a realizer F with modulus ψ,
then it has modulus $\Lambda(\eta,\theta,\psi)$.
b) If f has modulus λ,
then it has a realizer F with modulus $\Psi(\eta,\theta,\lambda)$
such that $\Lambda(\eta,\theta,\Psi(\eta,\theta,\lambda)) \approx \lambda$ and $\Psi(\eta,\theta,\Lambda(\eta,\theta,\psi)) \approx \psi$.

***In*tensional Linear *Main Thm*:** Let X and Y compact metric spaces, total representations $\xi{:}\mathcal{D}\to X$ and $\upsilon{:}\mathcal{E}\to Y$, and strictly increasing moduli of continuity μ,ν of ξ,υ and μ',ν' of ξ^{-1},υ^{-1}.
a) If $f{:}X\Rightarrow Y$ has a κ-continuous realizer, it is $\mu'\circ\kappa\circ\nu$-continuous.
b) Compact λ-continuous f has $\mu\circ\lambda\circ\nu'(n+O(1))$-contin. realizer.

Recap of Part III

- *Multi*functions
  - and (quantitative) continuity
- **Quantitative continuous selection theorem** for multifunctions between compact *ultra*metric spaces

  = domains of *generalized* representations
- *In*tensional" linear *Main Theorem*
- Hope for "*ex*tensional" quantitat. *Main Theorem*
- **Part IV:** (Relative) Complexity for Higher Types.

Computer Science of Numerics

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