§2 Computability over the Reals

a) Computing Real <u>Numbers</u>

- Three equivalent notions,
- counter/examples, oracle-computable reals

b) Computing Real <u>Sequences</u>

- semi-decidability / strong *un*decidability of Equality
- every computable sequence misses a computable Real

c) Computing Real <u>Functions</u>

- closure properties: composition, restriction, sequences
- necessarily continuous
- Computable Weierstrass Theorem
- quantitative continuity

§2 Computability over the Reals

d/e) Un/computability with Real Functions

- un/computable Derivative
- un/computable Wave Equation
- un/computable Root Finding

f) Multi-Functions & Enrichment

- generalized restriction, fundamental theorem of algebra
- real computation, fuzzy sign/Heaviside,
- Archimedian property, linear algebra, analytic functions

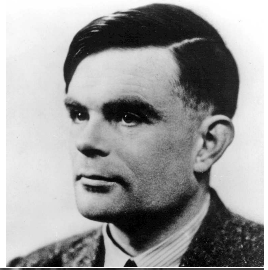
g) Computing Real Operators

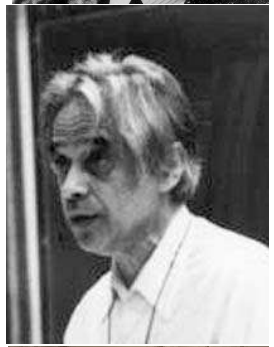
- Encoding continuous functions
- Encoding compact subsets
- Uniform computability
- Boolean Set Operations

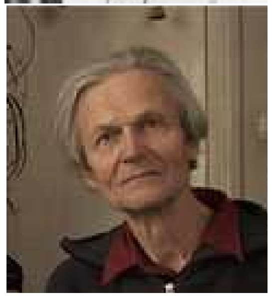
a) Computing Real Numbers

Theorem: For $r \in \mathbb{R}$, the following are equivalent:

Def: Call $r \in \mathbb{R}$ computable if







a) r has a decidable binary expansion

$\{ n : b_n = 1 \} \subseteq \mathbb{N}$ for $r = \sum_n b_n / 2^n$.

b) There exists an algorithm computing

a sequence $(a_n) \subseteq \mathbb{Z}$ with $|r - a_n/2^n| \leq 2^{-n}$.

c) There exist algorithms computing

three sequences $(a_m),(b_m),(c_m) \subseteq \mathbb{Z}$

with $|r - a_m/b_m| \leq 1/c_m \to 0$

Ernst Specker (1949): (c)^H $\Leftrightarrow$ (d) oracle

d) There is an algorithm computing $(q_n) \subseteq \mathbb{Q}$ s.t. $q_n \to r$.

$H = \{ \langle \mathcal{A}, \underline{x} \rangle$: algorithm $\mathcal{A}$ terminates on input $\underline{x}$ $\} \subseteq \mathbb{N}$

Proofs (Sketches)

Theorem: For $r\in\mathbb{R}$, the following are equivalent:

a) r has a decidable binary expansion
$\{\, n : b_n=1 \,\} \subseteq \mathbb{N}$ for $r = \sum_n b_n/2^n$.

b) There exists an algorithm computing
a sequence $(a_n)\subseteq\mathbb{Z}$ with $|r-a_n/2^n| \leq 2^{-n}$.

c) There exist algorithms computing
three sequences $(a_m),(b_m),(c_m)\subseteq\mathbb{Z}$
with $|r-a_m/b_m| \leq 1/c_m \to 0$

Lemma: For $r\in\mathbb{R}$ and $(a_n)\subseteq\mathbb{Z}$ with $|r-a_n/2^n| \leq 2^{-n}$,
$r<0 \;\Leftrightarrow\; \exists n: a_n<-1$ and $r>0 \;\Leftrightarrow\; \exists n: a_n>1$

Lemma: For $|x-y| \leq 1/2^{n+1}$, $a:=\lfloor y\cdot 2^n \rceil$ has $|x-a/2^n| \leq 2^{-n}$

Examples: Computable Reals

a) Every dyadic rational has two binary expansions

b) Every rational has a computable binary expansion

c) If a,b are computable, so are $a+b$, $a\cdot b$, $1/a$ $(a\neq 0)$

d) Fix $p\in\mathbb{R}[X]$. Then p's coefficients are computable

$\Leftrightarrow$ $p(x)$ is computable for all computable x.

e) Every algebraic number is computable; and so is π.

f) If x is computable, then so are $\exp(x)$, $\sin(x)$, $\log(x)$

g) Specker's sequence $\left(\sum_{m<j,\, t(m)<j} 2^{-m}\right)_j$ is "computable",

where $\{0,1,2,\ldots,\infty\} \ni t(\langle\mathcal{A},x\rangle):=$#steps $\mathcal{A}$ makes on x.

Compute r: on input $n\in\mathbb{N}$ output $a\in\mathbb{Z}$ st. $|r-a/2^n|\leq 2^{-n}$

Oracle-Computable Reals

$$\mathcal{P} = (\, x_j := 0\,,\,1 \mid x_j := x_i \pm x_k \mid x_j := x_i \div 2 \mid$$
$$x_j := \varphi(x_i) \mid \mathcal{P}\,;\,\mathcal{P} \mid \text{WHILE } x_j \text{ DO } P \text{ END}\,)$$

Fix some *arbitrary* total $\varphi:\mathbb{N}\to\mathbb{N}$

***Real* Limit Lemma:** If computable sequence (r_j) converges, then $r:=\lim_j r_j$ is computable with oracle H. And to every real r computable with oracle H, there is a computable sequence (r_j) with $r=\lim_j r_j$.

g) Specker's sequence $\left(\sum_{m<j,\, t(m)<j} 2^{-m}\right)_j$ is "computable",
where $\{0,1,2,\ldots,\infty\} \ni t(\langle\mathcal{A},x\rangle):=$#steps $\mathcal{A}$ makes on x.

oracle

Compute r: on input $n\in\mathbb{N}$ output $a\in\mathbb{Z}$ st. $|r-a/2^n|\le 2^{-n}$

b) Computing Real Sequences

Def: Compute sequence $(r_j)\subseteq\mathbb{R}$: on input $\langle n,j\rangle\in\mathbb{N}$ output some $a=a_{n,j}\in\mathbb{Z}$ with $|r_j-a/2^n|\leq 2^{-n}$.

Proposition: If (r_j) is a computable sequence s.t. $|r_j-r_i|\leq 2^{-j}+2^{-i}$, then $\lim_j r_j$ is a computable real.

Proposition: If (r_j) is a computable sequence, then $\{\ j : r_j\neq 0\ \}$ is semi-decidable.

In numerics, don't test for (in-)equality!

Examples: a) $1/j!$ is a computable sequence.

b) $\mathrm{cf}_H(j)\in\{0,1\}$ is an *un*computable sequence.

c) $r_j := 1/2^{t(j)}\in[0,1]$ is a computable sequence with $\{\ j : r_j\neq 0\ \} = H$, the Halting problem.

Any computable real sequence misses some computable real

Def: Compute sequence $(r_j)\subseteq\mathbb{R}$: on input $\langle n,j\rangle\in\mathbb{N}$ output some $a=a_{n,j}\in\mathbb{Z}$ with $|r_j-a/2^n|\leq 2^{-n}$.

Proof (Diagonalization):

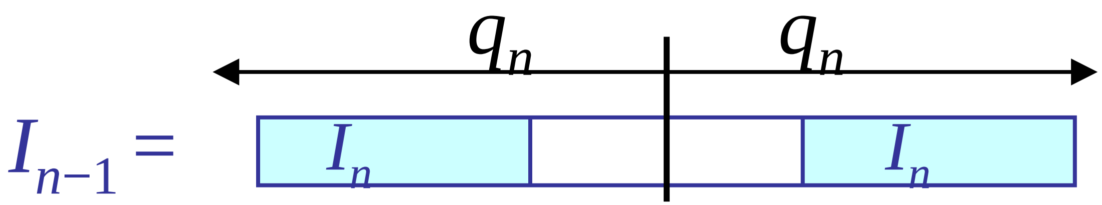
Consider 'diagonal' sequence $q_n := a_{2n+2,n}/2^{2n+1}\in\mathbb{Q}$.

Inductively define nested intervals $I_n\subseteq I_{n-1}$ of width $1/3^n$

such that $r_n\notin I_n$.

Hence $\{x\} = \bigcap_n I_n$

with <u>computable</u> $x\neq r_n$.



b) computing a sequence $(a_n)\subseteq\mathbb{Z}$ with $|r-a_n/2^n|\leq 2^{-n}$.

c) computing three sequences $(a_m),(b_m),(c_m)\subseteq\mathbb{Z}$
with $|r-a_m/b_m|\leq 1/c_m\to 0$

c) Computing Real Functions

Def: Compute sequence $(r_j)\subseteq\mathbb{R}$: on input $\langle n,j\rangle\in\mathbb{N}$ output some $a=a_{n,j}\in\mathbb{Z}$ with $|r_j-a/2^n|\leq 2^{-n}$.

Def: To compute $f:\subseteq\mathbb{R}\to\mathbb{R}$ means:

Convert *any* $(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\leq 2^{-m}$, $x\in\mathrm{dom}(f)$,

to some $(b_n)\subseteq\mathbb{Z}$ with $|y-b_n/2^n|\leq 2^{-n}$, $y=f(x)$.

Behave arbitrarily for $x\notin\mathrm{dom}(f)$ or $\exists m$: $|x-a_m/2^m|>2^{-m}$

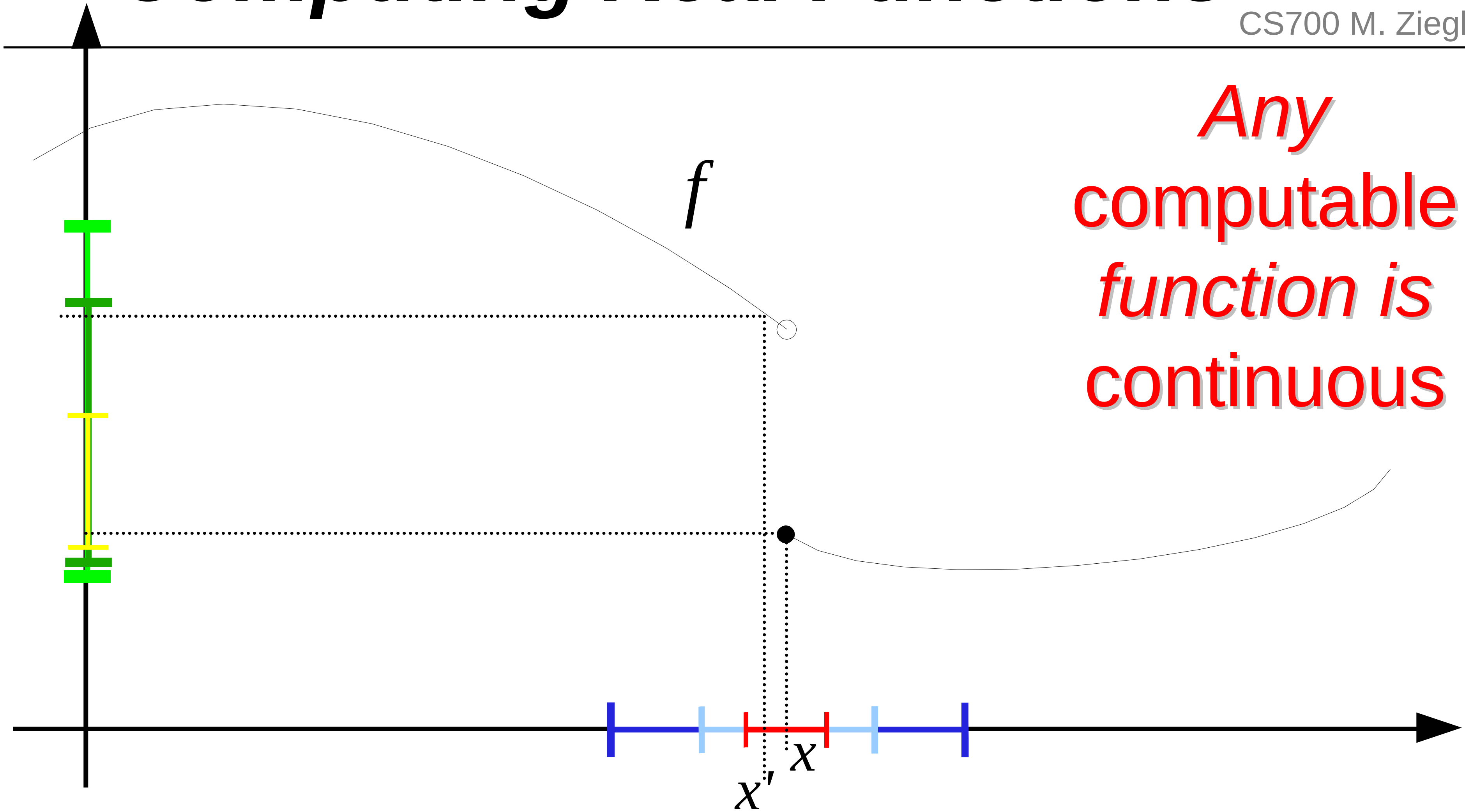
Lemma: a) If $f:\subseteq\mathbb{R}\to\mathbb{R}$ is computable and $(r_j)\subseteq\mathrm{dom}(f)$ are computable, then $f(r_j)$ is a computable sequence.

b) Computable functions are closed under composition

c) Any restriction of a comput. function is computable.

Compute r: on input $n\in\mathbb{N}$ output $a\in\mathbb{Z}$ st. $|r-a/2^n|\leq 2^{-n}$

Computing Real Functions



Any computable *function is* continuous

Def: Convert *any* $(a_m)\subseteq\mathbb{Z}$ with $|x\text{-}a_m/2^m|\leq 2^{-m}$,
to some $(b_n)\subseteq\mathbb{Z}$ with $|y\text{-}b_n/2^n|\leq 2^{-n}$, $y=f(x)$.

Computable Weierstrass Theorem

Theorem: For $f:[0,1]\to\mathbb{R}$ the following are equivalent:

a) There is an algorithm converting any $\underline{a}=(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\leq 2^{-m}$, to $(b_n)\in\mathbb{Z}$ with $|f(x)-b_n/2^n|\leq 2^{-n}$

b) There is an algorithm printing a sequence (of deg.s and coefficient lists of) $(P_n)\subseteq\mathbb{D}[X]$ with $\|f-P_n\|_\infty\leq 2^{-n}$

c) The real 'sequence' $f(q)$, $q\in\mathbb{D}\cap[0,1]$, is computable $\wedge$ f admits a computable modulus of (unif) continuity

$|x-y|\leq 2^{-\mu(n)} \Rightarrow |f(x)-f(y)|\leq 2^{-n}$

Proof: b $\Leftrightarrow$ c $\Rightarrow$ a $\Rightarrow$ c

$\mathbb{D}_n := \{\ a/2^n : a\in\mathbb{Z}\ \}, \qquad \mathbb{D} := \bigcup_n \mathbb{D}_n,$

Quantitative Continuity

Definition: Fix metric spaces (X,d) and (Y,e).

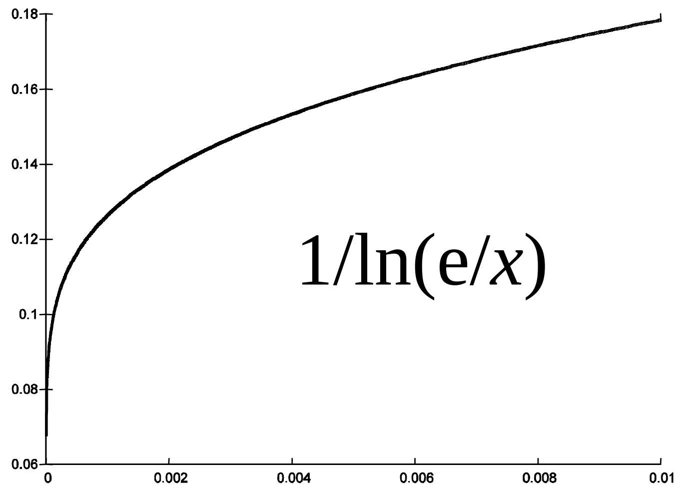
A modulus of continuity of $f:(X,d)\to(Y,e)$ is any $\mu:\mathbb{N}\to\mathbb{N}$ such that $d(x,x')\leq 2^{-\mu(n)}$ implies $e\big(f(x),f(x')\big)\leq 2^{-n}$

If $f:X\to Y$ has μ and $g:Y\to Z$ has ν, then $g\circ f$ has $\mu\circ\nu$.

Example: Lipschitz-continuous $\Leftrightarrow$ modulus $\mu(n)\leq n+O(1)$

b) Hölder-continuous $\Leftrightarrow$ modulus $\mu(n)\leq O(n)$

c) h: $[0;1]\ni x\to 1/\ln(e/x)\in[0;1]$ has (only) exponential modulus.



d) $h\circ h$: (only) *doubly* exponential modulus.

Examples of Computable Real Functions

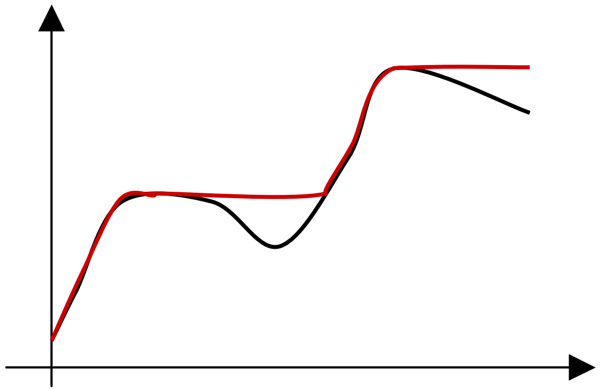
a) $+, -, \times, \div, \sqrt{}, \exp, \log_e, \sin, \cos$ are computable

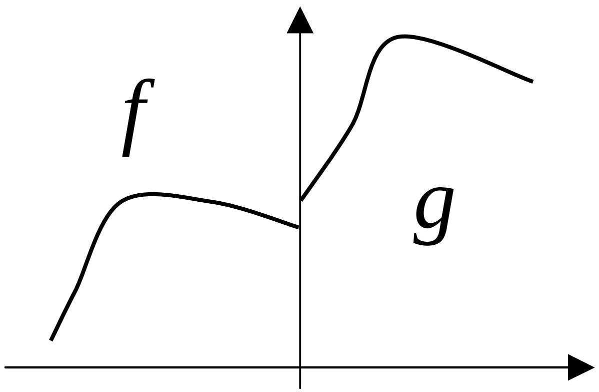
b) Let $f \in C[0,1]$ be computable. Then so are

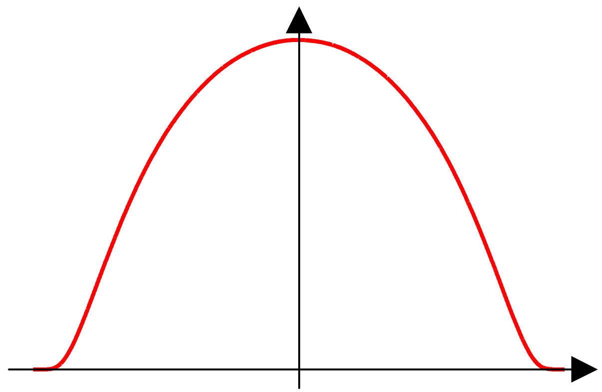
$\int f: x \to \int_0^x f(t)\,dt$ and $\max(f): x \to \max\{f(t): t \le x\}$.

c) For computable $f:[-1,0] \to \mathbb{R}$, $g:[0,1] \to \mathbb{R}$ with $f(0)=g(0)$, their join is computable.

d) C^∞ 'pulse' mollifier $\varphi(t) = \exp(-t^2/1-t^2)$ for $-1<t<1$,
$\varphi(t) = 0$ otherwise.







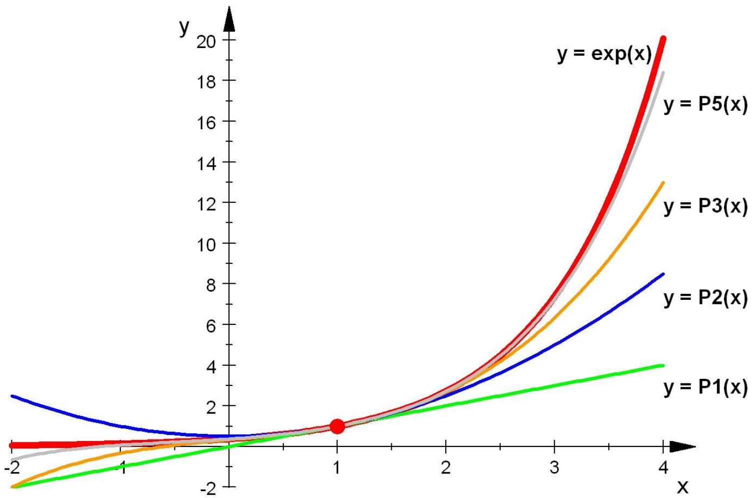
Example Proofs (Sketch)

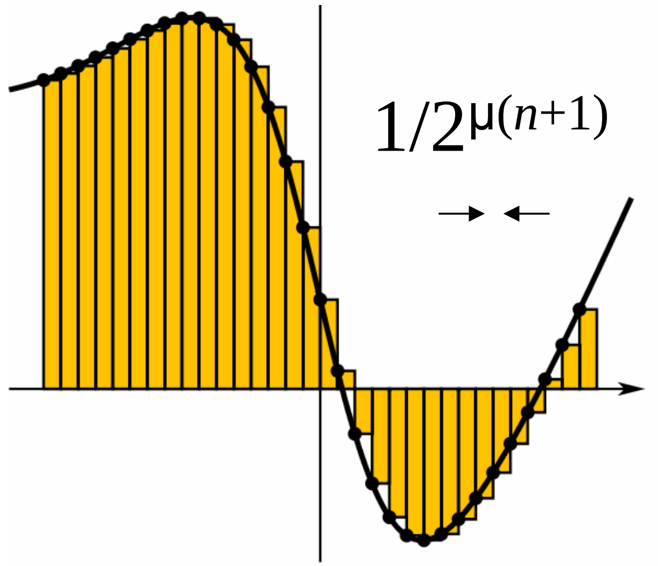
a) $\exp:\mathbb{R}\to\mathbb{R}$ is a computable function

$$\left|\exp(t) - (1+t+t^2/2+t^3/6+\ldots+t^n/n!)\right| \leq 1/2^n \text{ for } |t|\leq 1.$$

$$\exp(t+k) = \exp(t)\cdot\exp(1)\cdots\exp(1),\quad k\in\mathbb{N}$$

b) $f\in C[0,1]$ computable $\Rightarrow$ $\max(f):x\to\max\{f(t):t\leq x\}$





To **compute** $f:K\subseteq\mathbb{R}\to\mathbb{R}$: compute real 'sequence' $f(q)$, $q\in\mathbb{D}\cap K$; and compute *modulus* of continuity $\mu:\mathbb{N}\to\mathbb{N}$

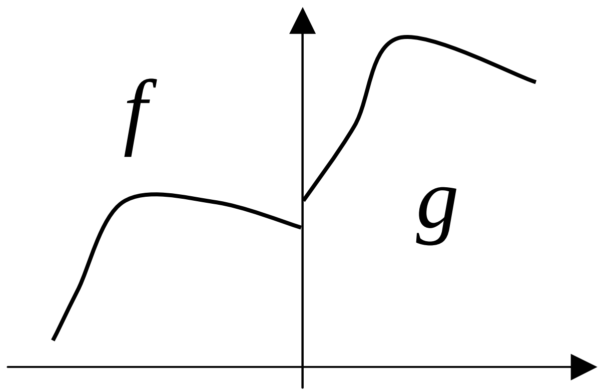
Example Proofs (continued)

a) $\exp:\mathbb{R}\to\mathbb{R}$ is a computable function

b) $f\in C[0,1]$ computable $\Rightarrow$ so are $\int f: x\to\int_0^x f(t)\,dt$

and $\max(f):x\to\max\{f(t):t\leq x\}$

c) For computable $f:[-1,0]\to\mathbb{R}$, $g:[0,1]\to\mathbb{R}$ with $f(0)=g(0)$, their join is computable.



To compute $f:K\subseteq\mathbb{R}\to\mathbb{R}$: compute real 'sequence' $f(q)$, $q\in\mathbb{D}\cap K$; and compute *modulus* of continuity $\mu:\mathbb{N}\to\mathbb{N}$

d) *Un*computability with Real Functions

- [Myhill'71]: **un**computable derivative
  - Sufficient condition for **computable** derivative
- [Pour-El&Richards'81]: **un**computable Wave Equation
  - [Weihrauch&Zhong'02] **computable** Wave Equation
- [Specker'59]: **un**computable argmin/root
  - Computable *Intermediate Value Theorem*
- Computable "singular" covering of all computable reals

*"Any computable function**al** is continuous!"*

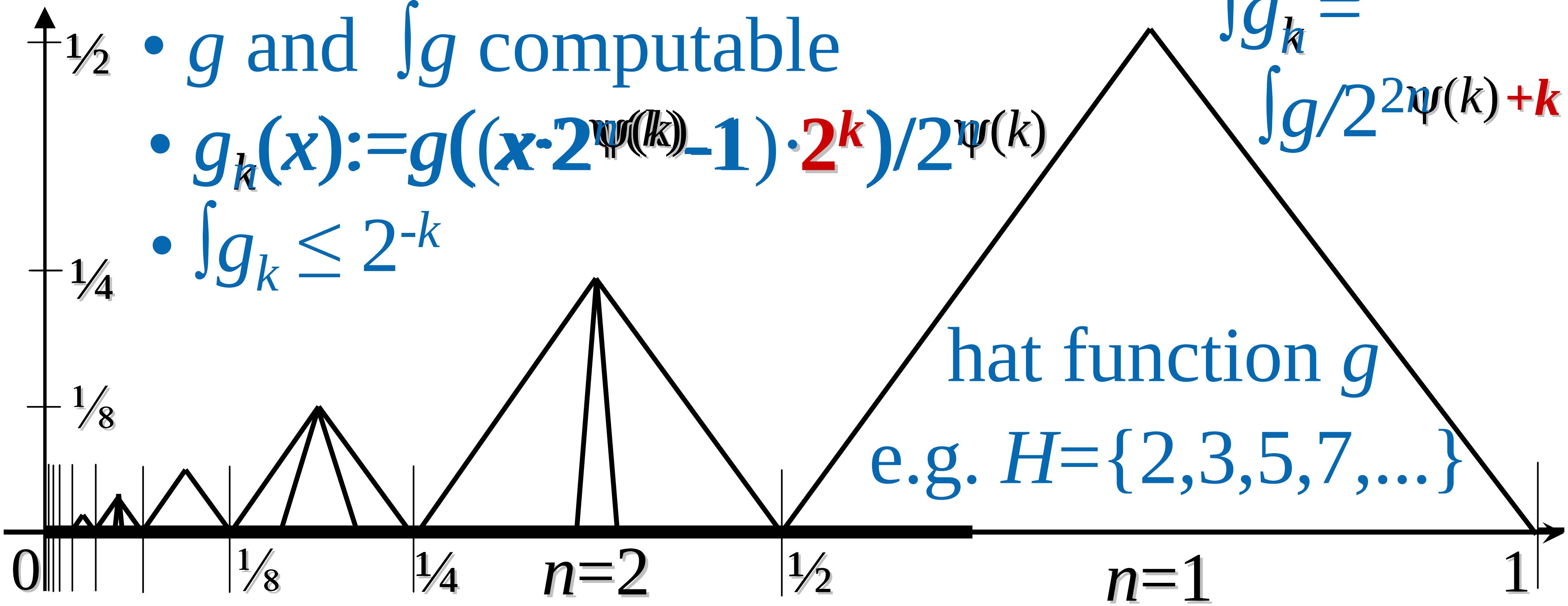
*Un*computable $\partial: C^1[0,1]\to C[0,1]$

CS700 M. Ziegler

Recall computable bijection $\psi:\mathbb{N}\to H$

- g and $\int g$ computable
- $g_k(x):=g((x\cdot 2^{\psi(k)}-1)\cdot 2^k)/2^{\psi(k)}$
- $\int g_k \le 2^{-k}$

$\int g_k = \int g/2^{2\psi(k)+k}$

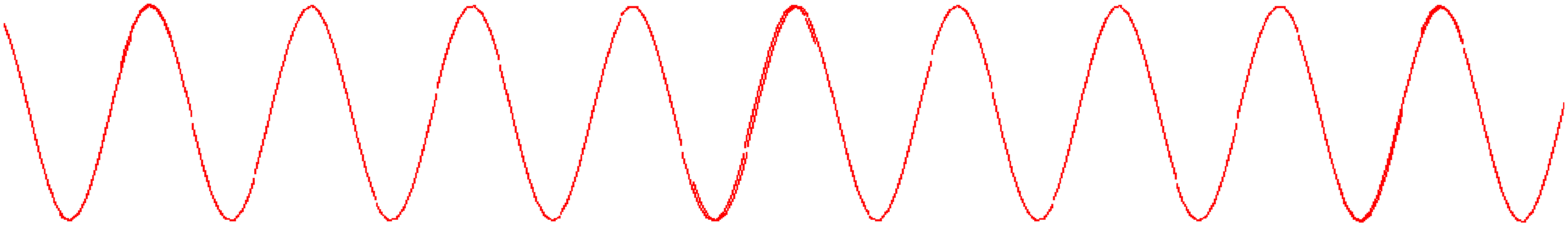


$h' := \sum_{k\in\mathbb{N}} g_k$ continuous, *un***computable**,

yet $h:=\int h' \in C^1[0;1]$ computable.

q.e.d.

e) *Computable* Derivative



Theorem: Suppose C^1 f:[0;1]→ℝ is computable.
Then f' is again computable iff
f' has a *computable* modulus of continuity μ'.

Proof: Given $x\in\mathbb{R}$, $n\in\mathbb{N}$, output $(f(x+\delta)-f(x))/\delta$. $\delta:=2^{-\mu'(n)}$

Then $f'(y) = (f(x+\delta)-f(x))/\delta$ for <u>some</u> $y\in[x,x+\delta]$:

Mean Value Theorem. By hypothesis, $|\,f'(y)-f'(x)\,| \leq 2^{-n}$.

Corollary: Suppose C^∞ f:[0;1]→ℝ is computable.
Then <u>each</u> derivative $f^{(k)}$, $k\in\mathbb{N}$, is again computable.

*Un*computable Wave Equation

Recall: computable $h \in C^1[0,1]$

with *un*computable $h'(1)$

3D Kirchhoff's formula:

$$u(t,\vec{x}) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\vec{y}-\vec{x}|=t} f(\vec{y})\, d\sigma(\vec{y})\right) + \frac{1}{4\pi t}\int_{|\vec{y}-\vec{x}|=t} g(\vec{y})\, d\sigma(\vec{y})$$

$$f(\vec{x}) := h(|\vec{x}|^2)$$

$$u(t,0) = \frac{d}{dt}\Big(h(t^2)\cdot t\Big) = h'(t^2)\cdot 2t^2 + h(t^2)$$

$\partial^2/\partial t^2\, u(\underline{x},t) = \Delta u(\underline{x},t)$, $u(\underline{x},0) = h(|\underline{x}|^2)$ $\partial/\partial t\, u(\underline{x},0) \equiv 0$

Computable Wave Equation

Example (spherical coord)**:** $f(r\cdot\sin\theta\cdot\cos\varphi, r\cdot\sin\theta\cdot\sin\varphi, r\cdot\cos\theta)$

$:= (r\text{-}1)\cdot(2\text{-}r)\cdot(\varphi\text{-}\pi/6)\cdot(\pi/4\text{-}\varphi)$ for $1 \leq r \leq 2$, $\pi/6 \leq \varphi \leq \pi/4$.

$:= 0$ otherwise

$$u(t,\vec{x}) \;=\; \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\vec{y}-\vec{x}|=t} f(\vec{y})\, d\sigma(\vec{y})\right)$$

$\Rightarrow u(1,0,0,0) \neq 0 = u(1,0,0,\varepsilon)$ $\forall \varepsilon \neq 0$: spatial discontinuity

[Weihrauch&Zhong'02] *Sobolev* space solution computable!

Mathematically well-known loss of regularity "*one derivative*":

$$u(t,0) \;=\; \frac{d}{dt}\Big(h(t^2)\cdot t\Big) \;=\; h'(t^2)\cdot 2t^2 + h(t^2)$$

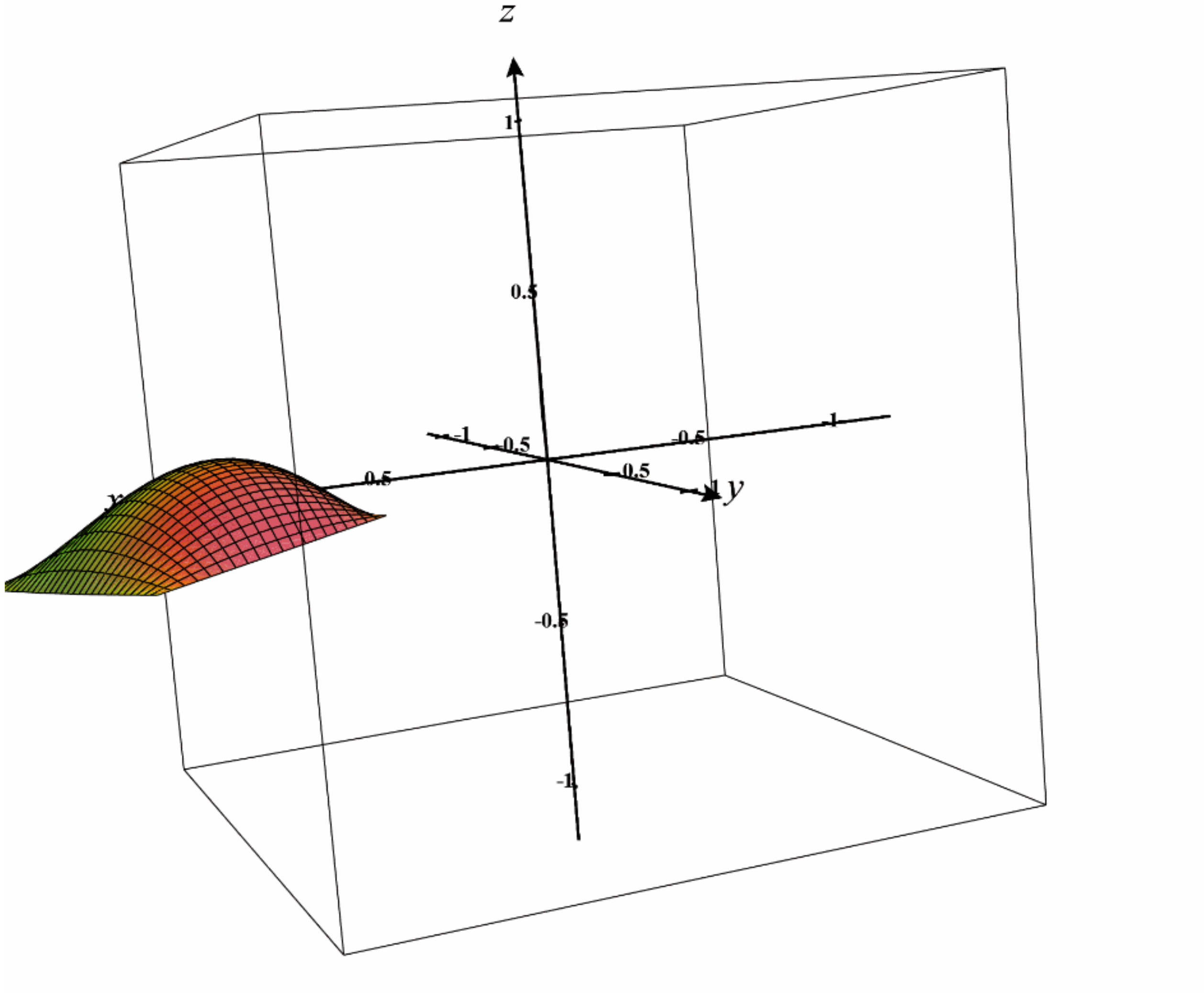
*"Any computable function**al** is continuous!"*

Discontinuous Wave Equation

Example (spherical coord)**:** $f(r\cdot\sin\theta\cdot\cos\varphi, r\cdot\sin\theta\cdot\sin\varphi, r\cdot\cos\theta)$

$:= (r-1)\cdot(2-r)\cdot(\varphi-\pi/6)\cdot(\pi/4-\varphi)$ for $1\le r\le 2$, $\pi/6\le\varphi\le\pi/4$.

$:= 0$



Computable `rootof`

Computable Intermediate Value Theorem:

Suppose $f:[0;1]\to[-1;1]$ is computable with $f(0)<0<f(1)$.
Then f has some computable root $x\in[0;1]$ with $f(x)=0$.

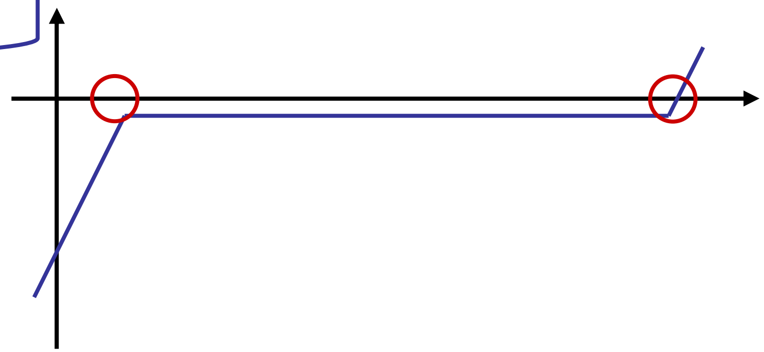
Proof (Bisection):

Initially $a:=0$, $b:=1$.

f has (at least one) root in $[a;b]$.

- If $f((a+b)/2)<0$ then let $a:=(a+b)/2$ and continue.
- If $f((a+b)/2)>0$ then let $b:=(a+b)/2$ and continue.
- If $f((a+b)/2)=0$ then return $(a+b)/2$.

n-th iteration: $|b-a| = 1/2^n$



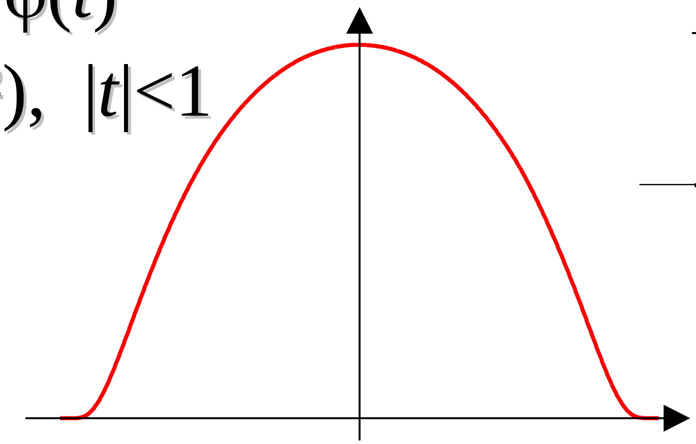
f has a root in $\mathbb{D}$:

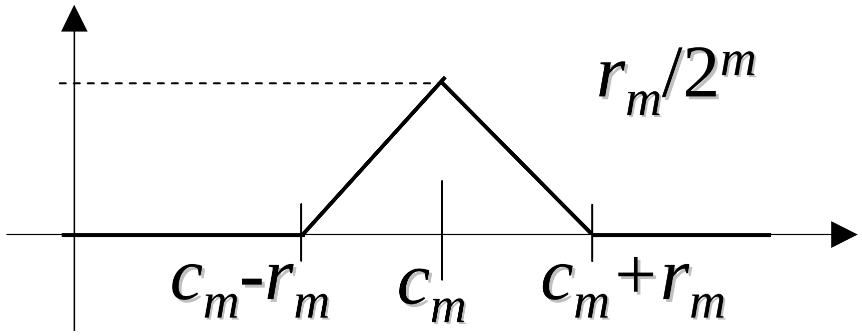
computable! (f is fixed!)

"Any computable functional is continuous!"

Computable Urysohn

C^∞ mollifier $\varphi(t)$
$= \exp(-t^2/1-t^2), \ |t|<1$
$= 0, \ |t| \geq 1$





Proof: Let $f(t) := \sum_m \varphi(r_m-|t-c_m|)/2^m$

Lemma: Let $(c_m)_m, (r_m)_m \subseteq \mathbb{D}$ be computable sequences.
There exists a computable C^∞ function $f:[0;1]\to[0;1]$
such that $f^{-1}[0] = [0;1]\setminus\bigcup_m (c_m-r_m,c_m+r_m)$.

*Un*computable `argmin`, `rootof`

Lemma: There exist computable sequences $(c_m)_m, (r_m)_m \subseteq \mathbb{D}$ such that $U := \bigcup_m (c_m - r_m, c_m + r_m)$ contains all computable reals in [0;1] and has measure $\leq$½.

approximating a root vs. *approximate* root

Corollary: There is a computable C^∞ $f:[0;1]\to[0;1]$ such that $f^{-1}[0]$ has measure $\geq$½ but contains no computable real number.

Lemma: Let $(c_m)_m, (r_m)_m \subseteq \mathbb{D}$ be computable sequences. There exists a computable C^∞ function $f:[0;1]\to[0;1]$ such that $f^{-1}[0] = [0;1] \setminus \bigcup_m (c_m - r_m, c_m + r_m)$.

"A countable real set has measure 0"

Lemma: There exist sequences $(c_m)_m, (r_m)_m \subseteq \mathbb{D}$ such that $U := \bigcup_m (c_m - r_m, c_m + r_m)$ covers any fixed countable subset of $[0;1]$ and has measure $\leq$½.

$\mathcal{P}$ computes $r \in \mathbb{R}$ iff prints sequence $a_n \subseteq \mathbb{Z}$ with $|a_n/2^n - a_m/2^m| \leq 2^{-n} + 2^{-m}$

Proof idea (diagonalize against <u>all</u> $\mathcal{P}$):

Simulate program $\mathcal{P}$ until it outputs $(a_0, a_1, \ldots a_{\langle\mathcal{P}\rangle+4}) \in \mathbb{Z}^*$ s.t. $0 \leq a_n \leq 2^n$, $|a_n/2^n - a_m/2^m| \leq 2^{-n} + 2^{-m}$ $\forall n,m \leq \langle\mathcal{P}\rangle+4$ and let $c_{\langle\mathcal{P}\rangle} := a_{\langle\mathcal{P}\rangle+4}/2^{\langle\mathcal{P}\rangle+4}$ and $r_{\langle\mathcal{P}\rangle} := 1/2^{\langle\mathcal{P}\rangle+3}$.

U has measure $\leq \sum_{\langle\mathcal{P}\rangle} 2r_{\langle\mathcal{P}\rangle} =$ ½.

What if $\mathcal{P}$ does not produce *in*finite output?

d) *Un*/computability with Real Functions

- [Myhill'71]: **un**computable derivative
  - Sufficient condition for **computable** derivative
- [Pour-El&Richards'81]: **un**computable Wave Equation
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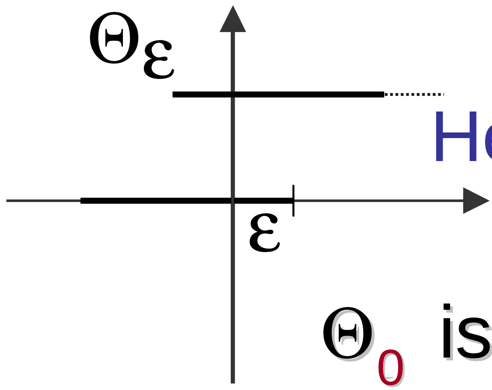
*"Any computable functio**nal** is continuous!"*

f) *Multi*functions

A partial *multi*function $G:\subseteq X\rightrightarrows Y$ is a relation $G\subseteq X\times Y$ / *set*function $G:X\to\mathcal{P}(Y)$

Any computable *function is* continuous

- Aka non-extensional "functions"
- Unavoidable in real computation!



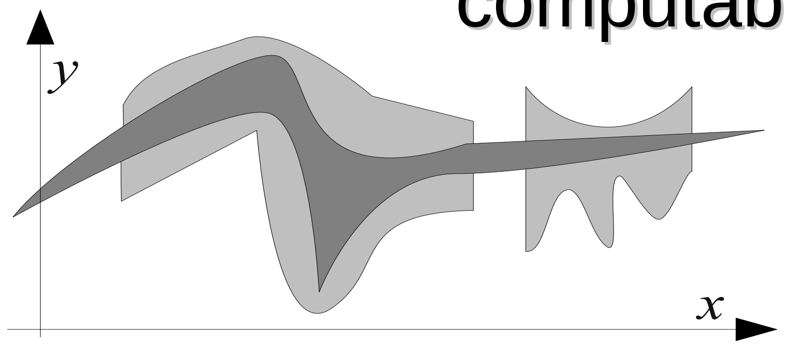
Heaviside

Θ_0 is *not* computable.

Restriction $F\sqsubseteq G$: smaller domain and/or *larger* range(s).

"easier"

Function problem:
Input x, output $y=G(x)$;
not necessarily *all* $y\in G(x)$



Computable *Multi*functions

A partial *multi*function $G :\subseteq X \rightrightarrows Y$ is a relation $G \subseteq X \times Y$ / ~~setfunction $G: X \to \mathcal{P}(Y)$~~

Any computable *function is* continuous

Archimedian Property of the Reals:

There ***is*** a computable ***multi-*** function $f: \mathbb{R} \rightrightarrows \mathbb{Z}$ with $f(r) \geq r$.

Fundamental Theorem of Algebra:

Given $a_0, \ldots a_{d-1} \in \mathbb{C}$, return roots $x_1, \ldots x_d \in \mathbb{C}$ of monic $a_0 + a_1 \cdot X + \ldots + a_{d-1} \cdot X^{d-1} + X^d \in \mathbb{C}[X]$

in which order??

incl. multiplicities

Def: Compute $f :\subseteq \mathbb{R} \rightrightarrows \mathbb{R}$:

Convert *any* $(a_m) \subseteq \mathbb{Z}$ with $|x - a_m/2^m| \leq 2^{-m}$, $x \in \mathrm{dom}(f)$, to some $(b_n) \subseteq \mathbb{Z}$ with $|y - b_n/2^n| \leq 2^{-n}$, $y \in f(x)$.

Enrichment in Linear Algebra

KAIST
CS700 M. Ziegler

- rank: $\mathbb{R}^{d\times d} \to \mathbb{N}$ *dis*continuous, uncomputable
  - Gauss' Algorithm: *pivoting* = test for in/equality
  - dimension/basis of kernel/range, eigenvectors: *un*computable
- rank: $\subseteq \mathbb{R}^{d\times d} \times \mathbb{N} \ni (A, \boldsymbol{r}=\text{rank}(A)) \to \text{rank}(A) \in \mathbb{N}$ trivial
  - kernelbasis: $(A, \boldsymbol{r}=\text{rank}(A)) \Rightarrow \mathbb{R}^{d\times \boldsymbol{r}}$ Computable!

    [Algorithm: ***r*** rounds of LUPQ decomposition with full pivoting…]
  - eigenbasis: $\{(A,\delta)$: symmetric $A \in \mathbb{R}^{d\times(d-1)/2}$ has exactly $\delta \in \mathbb{N}$ distinct eigenvalues$\} \Rightarrow \mathbb{R}^{d\times \boldsymbol{d}}$ Computable!

    $\delta := \text{Card}\, \sigma(A)$

"*Enrichment*": G.Kreisel&A.Macintyre p.238/239 in "The L.E.J. Brouwer Centenary Symposium"1982 (Troelstra&van Dalen edt.s)

```
REAL **diagonalize(int d,REAL **matrix);
                   int nDistinctEValues);
```

canonical declaration

More Examples of *Enrichment*

Recall: Suppose C^2 $f:[0;1]\to\mathbb{R}$ is computable.

Then f' is again computable **!**

But must "know" some bound $B\in\mathbb{N}$ on f'' **!**

Recall: Computable Intermediate Value Theorem

Suppose $f:[0;1]\to[-1;1]$ is computable with $f(0)<0<f(1)$.

Then f has some computable root $x\in[0;1]$ with $f(x)=0$.

Enrichment: root$\in\mathbb{D}$ or "promise": no root$\in\mathbb{D}$

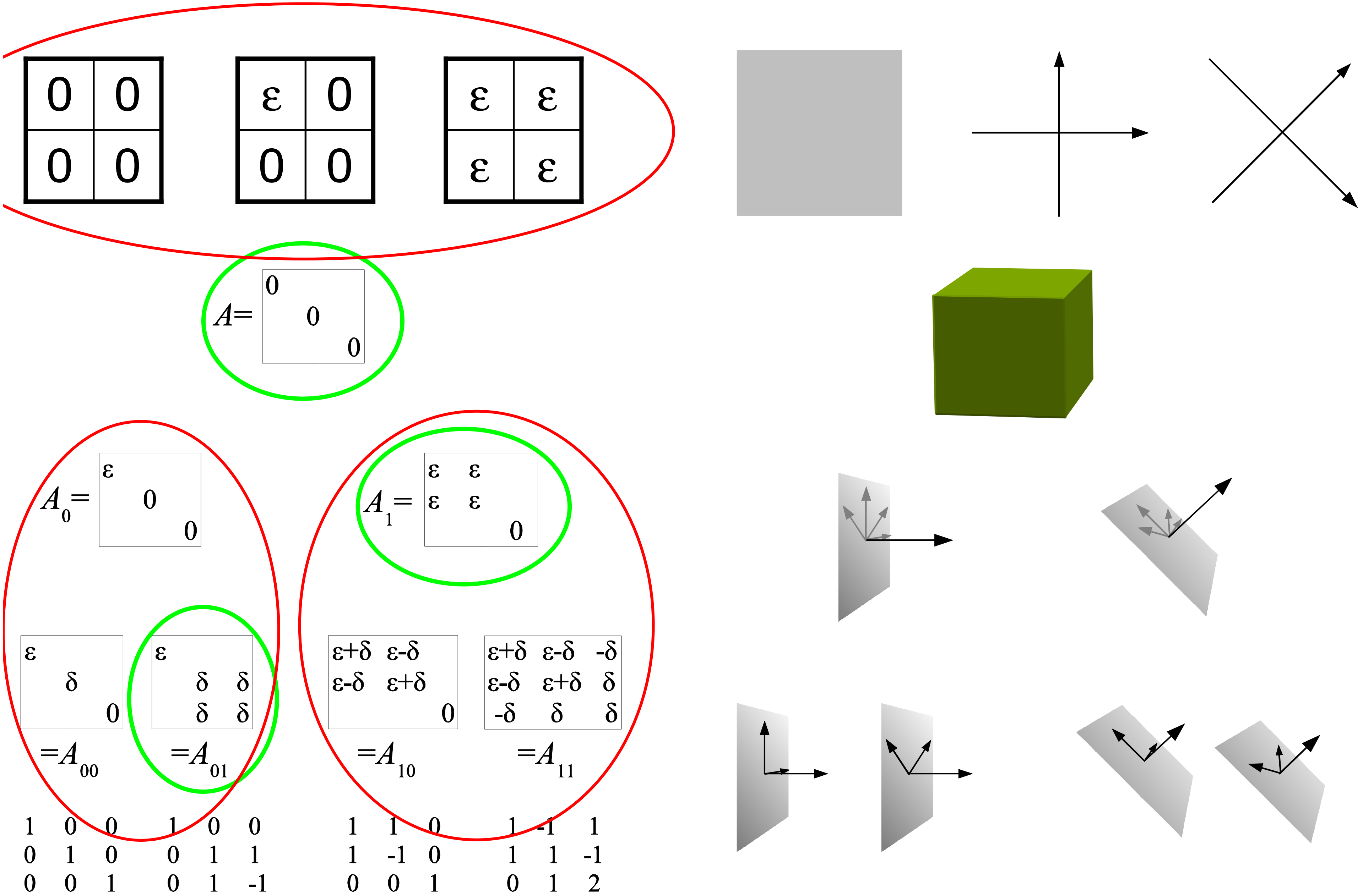
Consider power series $f(z)=\sum_m c_m\cdot z^m$, $c_m\in\mathbb{C}$ computable.

Radius of converg. $0<R=1/\limsup_m |c_m|^{1/m}$.

Fix any $r<R$. $\exists B\in\mathbb{N}\ \forall m$: $|c_m|\le B/r^m$.

$\Rightarrow$ computable tail bound $|\sum_{m>M} c_m\cdot z^m|\le$ geometric series.

(d-1)-fold Advice does not suffice for d×d Symmetric Matrix Diagonalization



g) Computing Real Operators

Compute $r\in\mathbb{R}$: <u>print</u> sequence $(a_n)\subseteq\mathbb{Z}$ st. $|r-a_n/2^n|\leq 2^{-n}$

Recall: To compute $f:K\subseteq\mathbb{R}\to\mathbb{R}$ means:

<u>Convert</u> *any* $(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\leq 2^{-m}$, $x\in\mathrm{dom}(f)$,

to some $(b_n)\subseteq\mathbb{Z}$ with $|y-b_n/2^n|\leq 2^{-n}$, $y=f(x)$.

Equivalent (Weierstraß): <u>print</u> a sequence (of degrees and coefficient lists of) $(P_n)\subseteq\mathbb{D}[X]$ with $\|f-P_n\|_\infty\leq 2^{-n}$

Definition: To compute $\Xi:\subseteq C(K)\to C(K')$ means:

<u>Convert</u> *any* $(P_m)\subseteq\mathbb{D}[X]$ with $\|f-P_m\|_\infty\leq 2^{-m}$, $f\in\mathrm{dom}(\Xi)$,

to some $(Q_n)\subseteq\mathbb{D}[X]$ with $\|g-Q_n\|_\infty\leq 2^{-n}$, $g=\Xi(f)$.

Any computable function**al**/operator is continuous!

***Non*/uniform *Un*/computability**

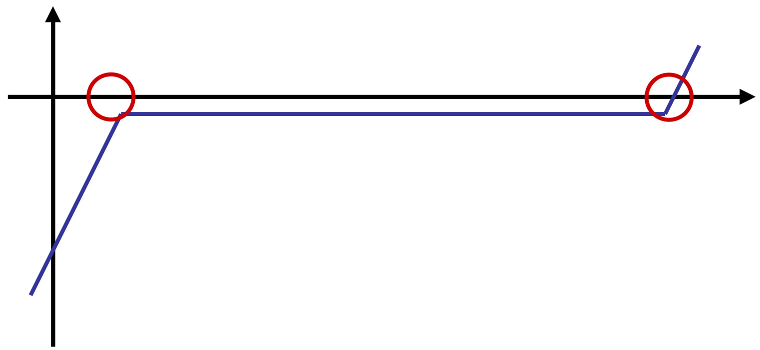
Non-uniform computability: "If f is computable, so is $\Lambda(f)$".

Stronger *uniform* computability: "$\Lambda: f \to \Lambda(f)$ is computable"

Applies also to *un*computable f, requires way of encoding f!

Uniformly computable

$\Rightarrow$ continuous.



Discontinuous:

- $\partial : C^1[0,1] \to C[0,1]$
- `rootof`

Definition: To compute $\Xi :\subseteq C(K) \to C(K')$ means:

Convert *any* $(P_m) \subseteq \mathbb{D}[X]$ with $\|f - P_m\|_\infty \le 2^{-m}$, $f \in \mathrm{dom}(\Xi)$,

to some $(Q_n) \subseteq \mathbb{D}[X]$ with $\|g - Q_n\|_\infty \le 2^{-n}$, $g = \Xi(f)$.

Any computable function**al**/operator is continuous!

Uniformly Computable Op.s

a) Pointwise addition, multiplication are computable.

b) Composition $(f,g) \to g \circ f$ is computable. So is **join**.

c) The operators $\int$ and $\max()$ are computable, where

$$\int f: x \to \int^x f(t)\, dt \quad \text{and} \quad \max(f): x \to \max\{f(t): t \leq x\}.$$

d) *Un*computable: • $\partial: C^1[0,1] \to C[0,1]$ • `rootof`

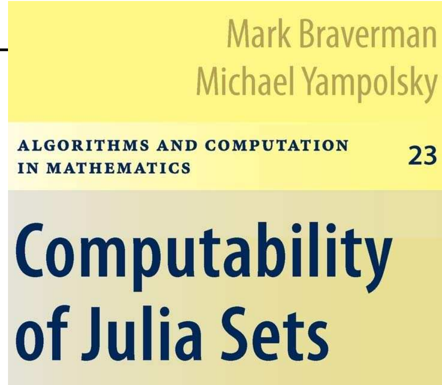
Definition: To compute $\Xi :\subseteq C(K) \to C(K')$ means:

Convert *any* $(P_m) \subseteq \mathbb{D}[X]$ with $\|f - P_m\|_\infty \leq 2^{-m}$, $f \in \mathrm{dom}(\Xi)$,

to some $(Q_n) \subseteq \mathbb{D}[X]$ with $\|g - Q_n\|_\infty \leq 2^{-n}$, $g = \Xi(f)$.

***Non*-uniform:** "Fix computable f,g. Then $f+g$ is computable"

Encoding Compact Subsets

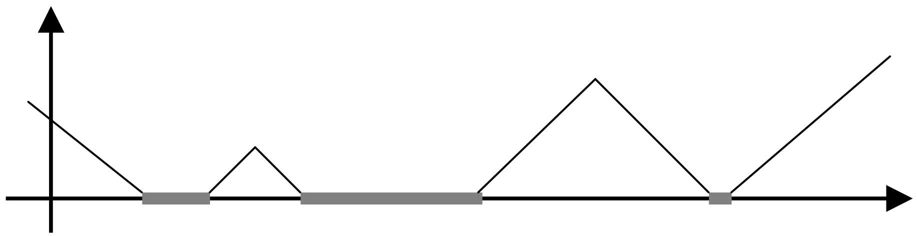


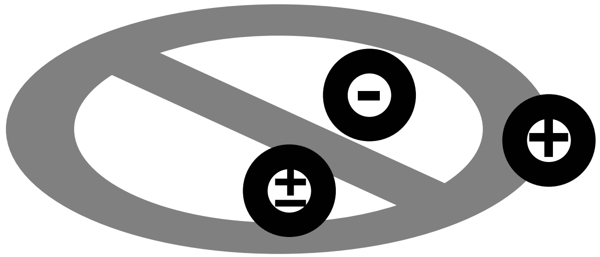
Examples/applications: Computing Fractals

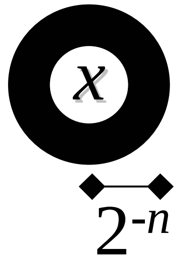
Def: Call $A \in \mathcal{K}(X)$ computable if

a) the distance function d_A is computable

b) the soft characteristic *multi*function 2_A is computable







(X,d) metric space

Soft characteristic *multi*function of $A \subseteq X$:

$2_A(x,n) = +$ if $d_A(x) \leq 2^{-n}$, $2_A(x,n) = -$ if $d_A(x) \geq 2^{-n-1}$

Distance function of $A \subseteq X$: $d_A : X \ni \underline{x} \to \inf\{\, d(\underline{x},\underline{a}) : \underline{a} \in A \,\} \in \mathbb{R}$

$\mathcal{K}(X)$ = { non-empty compact subsets of topolog. space X }

Un/computable Set Operations

$d_A : X \ni x \to \inf\{\, d(x,a) : a \in A \,\} \in \mathbb{R}$

$2_A(x,n) = +$ if $d_A(x) \le 2^{-n}$,

$2_A(x,n) = -$ if $d_A(x) \ge 2^{-n-1}$

Def: Call $A \in \mathcal{K}(X)$ computable

a) if the distance function d_A is computable

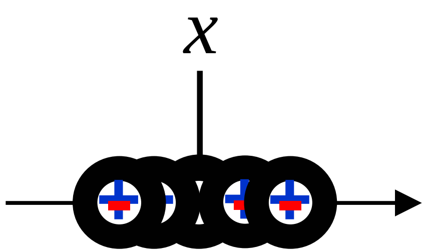
b) if the soft characteristic *multi*function 2_A is computable

Theorem: (a) and (b) are equivalent [even uniformly].

Proof, a) $\Rightarrow$ b): immediate.

b) $\Rightarrow$ a): scan grid of width 2^{-n-1}

$d_A(x) = ?\ \pm 2^{-n}$



Theorem (Boolean operations on compact sets):

a) $\cup$ is computable: √

b) $\cap$ is *un*computable:

§2 Computability over the Reals

a) Computing Real Numbers

- Three equivalent notions,
- counter/examples, oracle-computable reals

b) Computing Real Sequences

- semi-decidability / strong *un*decidability of Equality
- every computable sequence misses a computable Real

c) Computing Real Functions

- closure properties: composition, restriction, sequences
- necessarily continuous
- Computable Weierstrass Theorem
- quantitative continuity

d/e) Un/computability with Real Functions

- un/computable Derivative
- un/computable Wave Equation
- un/computable Root Finding

f) Multi-Functions & Enrichment

- generalized restriction, fundamental theorem of algebra
- real computability, fuzzy sign, Archimedian property
- linear algebra, analytic functions

g) Computing Real Operators

- Encoding continuous functions
- Encoding compact subsets
- Uniform computability
- Boolean Set Operations