§3 Computability on Topological Spaces

a) Basic Spaces

- Cantor/Baire Space, Computation
- Cost, Continuity, Compactness

b) Representations

- Definition, Real Examples revisited
- Realizers, Multi/Functions between Represented Spaces
- (Continuous) Reduction between Representations
- Standard/Admissible representations; *Main Theorem*
- Sequences, Continuous Functions, Compact Subsets

a) Basic Spaces

Classical/discrete/*countable* data processing: $\{\mathbf{0},\mathbf{1}\}^*, \mathbb{Z}^*$

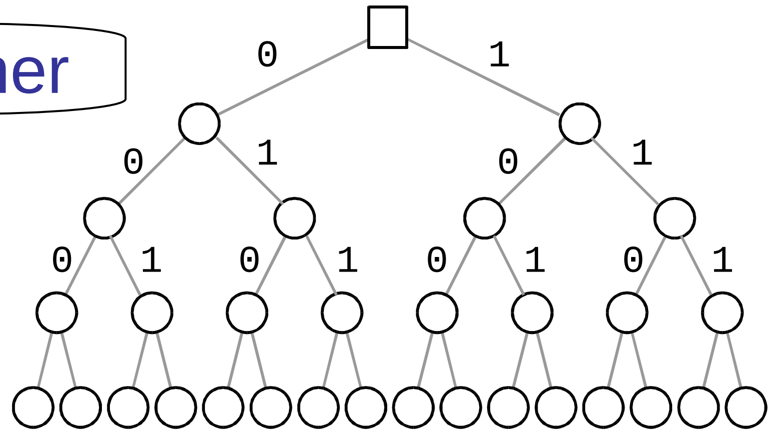
Input/process/output of (*finite* sequences of) bits or integers.

Other data (e.g. graphs) encode over *finite* no. bits/integers.

Universe of ***continuum* cardinality**, such as $\mathbb{R}, C(K), \mathcal{K}(X)$ **:**

Encode over ***in**finite* sequences of bits/integers.

Re/en-code one over the other



Cantor space $\mathcal{C} = \{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$

Baire space $\mathcal{B} = \mathbb{Z}^{\mathbb{N}}$

equipped with *ultra*metric

$D(\underline{u},\underline{v}) = 2^{-\min\{n:u_n \neq v_n\}}$

$D(\underline{u},\underline{v}) \leq \max\{D(\underline{u},\underline{w}),D(\underline{w},\underline{v})\}$

in binary

Def: Output $\underline{u}\in \mathcal{C}$ or $\underline{u}\in \mathcal{B}$: Print the sequence $u_0,u_1,u_2,\ldots$

Output in time $t:\mathbb{N}\rightarrow\mathbb{N}$: u_n appears after $\leq t(n)$ steps.

Computing on Basic Spaces

Def: Output $\underline{u}\in\mathcal{C}$ or $\underline{u}\in\mathcal{B}$: Print the sequence $u_0,u_1,u_2,\ldots$

Output in time $t:\mathbb{N}\to\mathbb{N}$: u_n appears after $\leq t(n)$ steps.

Def: Compute $F:\subseteq\mathcal{B}\to\mathcal{B}$: On input $\underline{u}\in\mathrm{dom}(F)$, output $F(\underline{u})$.

Behave arbitrarily on other inputs

regardless of $\underline{u}\in\mathrm{dom}(F)$

Compute in time $t:\mathbb{N}\to\mathbb{N}$: $F(u)_n$ appears after $\leq t(n)$ steps.

Example: $F(\underline{u}) =$ **111..** if #initial **0**s in $\underline{u}$ is **odd**
$F(\underline{u}) =$ **000..** if #initial **0**s in $\underline{u}$ is **even**

$t_{\mathcal{A}}(\underline{u},n)$:= #steps algor. $\mathcal{A}$ makes on input $\underline{u}$ until n-th output.

***Main* Lemma: a)** Every computable $F:\subseteq\mathcal{B}\to\mathcal{B}$ is continuous.

b) F computable in time t $\Rightarrow$ t is a modulus of continuity of F.

c) partial $\underline{u}\to t_{\mathcal{A}}(\underline{u},n)$ is locally constant/continuous.

d) $\mathrm{dom}(F)$ compact, $\mathcal{A}$ computes F $\Rightarrow$ has time bound $t=t(n)$

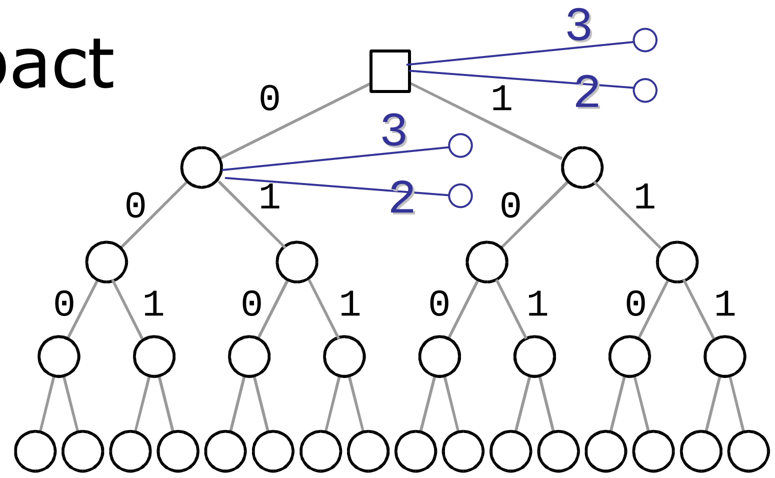
Compactness in Basic Spaces

Reminder a) Cantor space $C = \{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$ is compact.

b) A subset X of a compact set is compact iff X is closed.

c) Baire space $\mathcal{B} = \mathbb{Z}^{\mathbb{N}}$ is *not* compact.

König's Lemma: $X \subseteq \mathbb{Z}^{\mathbb{N}}$ is compact iff it is closed and the set
$X^* := \{ \bar{a} \in \mathbb{Z}^* \mid \exists \underline{b} \in \mathbb{Z}^{\mathbb{N}} : \bar{a}\,\underline{b} \in X \}$
of finite initial segments is finitely branching.



***Main* Lemma: a)** Every computable $F :\subseteq \mathcal{B} \to \mathcal{B}$ is continuous.

b) F computable in time $t \Rightarrow t$ is a modulus of continuity of F.

c) partial $\underline{u} \to t_{\mathcal{A}}(\underline{u}, n)$ is locally constant/continuous.

d) $\mathrm{dom}(F)$ compact, $\mathcal{A}$ computes $F \Rightarrow$ has time bound $t = t(n)$

b) Representations

binary: $r = \sum_n c_n 2^{-n}$, $\hat{c}=(c_n) \in C$ $\beta :\subseteq C \to [0;1]$

rational: $|r - a_{2n} / a_{2n+1}| \leq 1/n$ $\rho :\subseteq C \to [0;1]$

dyadic: $|r - a_n / 2^n| \leq 2^{-n}$ $(\mathrm{bin}(a_n)_n) \in C$ $\delta :\subseteq C \to [0;1]$

Recall equivalence: r has (a) decidable binary expansion

b) computable sequence $(a_n) \subseteq \mathbb{Z}$ with $|r - a_{2n}/a_{2n+1}| \leq 1/n$

c) computable sequence $(a_n) \subseteq \mathbb{Z}$ with $|r - a_n/2^n| \leq 2^{-n}$.

Def: A representation of a set X is a surjective partial mapping $\xi :\to X$.	A ξ-name of $x \in X$ is any $\underline{u}$ with $\xi(\underline{u}) = x$.

(Computing) Multi/Functions between Represented Spaces

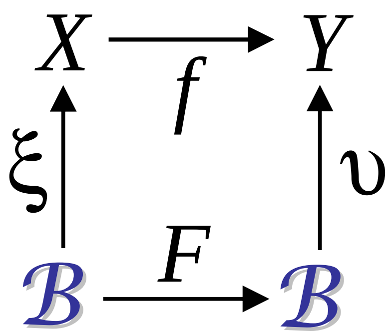
binary: $r = \sum_n c_n 2^{-n}, \ \hat{c}=(c_n) \in C$ $\quad \beta :\subseteq C \to [0;1]$

rational: $|r - a_{2n}/a_{2n+1}| \leq 1/n$ $\quad \rho :\subseteq C \to [0;1]$

dyadic: $|r - a_n/2^n| \leq 2^{-n}$ $\quad (\text{bin}(a_n)_n) \in C$ $\quad \delta :\subseteq C \to [0;1]$

Observe: For $\underline{u}$ a ξ-name of $x \in X$ and F a (ξ,υ)-realizer of $f:X \rightrightarrows Y$, $F(\underline{u})$ is a υ-name of $y \in f(x)$.

For F a (ξ,υ)-realizer of $f:X \rightrightarrows Y$ and G a (υ,ζ)-realizer of $g:Y \rightrightarrows Z$, $G \circ F$ is a (ξ,ζ)-realizer of $g \circ f$.



Computing $f:X \rightrightarrows Y$ means to compute a (ξ,υ)-realizer.

restriction

A (ξ,υ)-realizer of $f:X \rightrightarrows Y$ is a $F:\text{dom}(\xi) \to \text{dom}(\upsilon)$ s.t. $f \circ \xi \sqsubseteq \upsilon \circ F$.

Def: A representation of a set X is a surjective partial mapping $\xi :\to X$.

A ξ-name of $x \in X$ is any $\underline{u}$ with $\xi(\underline{u}) = x$.

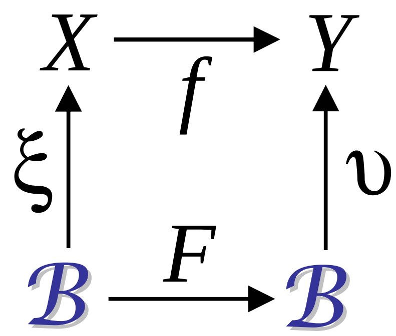
Reduction between Representations

binary: $r = \sum_n c_n 2^{-n}$, $\hat{c}=(c_n) \in C$ $\quad \beta:\subseteq C \to [0;1]$

rational: $|r - a_{2n}/a_{2n+1}| \le 1/n$ $\quad \rho:\subseteq C \to [0;1]$

dyadic: $|r - a_n/2^n| \le 2^{-n}$ $\quad \delta:\subseteq C \to [0;1]$

Examples: $\beta \preceq \rho$, $\beta \preceq \delta$, $\delta \preceq \rho$, $\rho \preceq \delta$, $\rho \not\preceq \beta$, $\delta \not\preceq \beta$.



Def: Continuous reduction $\xi' \preceq \xi$ means a (ξ',ξ)-realizer of $\mathrm{id}:X \to X$.

transitive

Computing $f:X \rightrightarrows Y$ means to compute a (ξ,υ)-realizer.

Reduction $\xi' \preceq \xi$ means cont. $F:\mathrm{dom}(\xi') \to \mathrm{dom}(\xi)$ st. $\xi' \sqsubseteq \xi \circ F$

Def: A representation of a set X is a surjective partial mapping $\xi:\to X$.

A ξ-name of $x \in X$ is any $\underline{u}$ with $\xi(\underline{u})=x$.

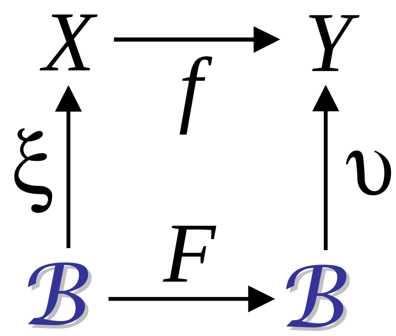
Domains of Representations

binary: $r = \sum_n c_n 2^{-n}$, $\hat{c}=(c_n) \in C$ $\qquad \beta :\subseteq C \to [0;1]$

rational: $|r - a_{2n} / a_{2n+1}| \leq 1/n$ $\qquad \rho :\subseteq C \to [0;1]$

dyadic: $|r - a_n / 2^n| \leq 2^{-n}$ $\qquad \delta :\subseteq C \to [0;1]$

Examples: $\beta \preceq \rho$, $\beta \preceq \delta$, $\delta \preceq \rho$, $\rho \preceq \delta$, $\rho \not\preceq \beta$, $\delta \not\preceq \beta$.



Examples: dom(ρ) is not compact,
dom(β), dom(δ) are compact. König's Lemma

Recall: dom(F) compact, $\mathcal{A}$ computes $F \Rightarrow$ has time bound t

König's Lemma: $X \subseteq \mathbb{Z}^{\mathbb{N}}$ is compact iff it is closed and the set $X^* := \{ \bar{a} \in \mathbb{Z}^* \mid \exists \underline{b} \in \mathbb{Z}^{\mathbb{N}} : \bar{a}\, \underline{b} \in X \}$ of finite initial segments is finitely branching.

Admissible Representations

binary: $r = \sum_n c_n 2^{-n}$, $\hat{c}=(c_n) \in C$ $\beta :\subseteq C \to [0;1]$

rational: $|r - a_{2n}/a_{2n+1}| \leq 1/n$ $\rho :\subseteq C \to [0;1]$

dyadic: $|r - a_n/2^n| \leq 2^{-n}$ $\delta :\subseteq C \to [0;1]$

Examples: $\beta \preceq \rho$, $\beta \preceq \delta$, $\delta \preceq \rho$, $\rho \preceq \delta$, $\rho \not\preceq \beta$, $\delta \not\preceq \beta$.

Examples: dom(β), dom(δ) compact, dom(ρ) not compact

Examples: β is not admissible. ρ and δ are admissible.

Def: Representation ξ of X is admissible if (i) is continuous and (ii) every continuous representation ξ' of X has: $\xi' \preceq \xi$.

Reduction $\xi' \preceq \xi$ means cont. F:dom(ξ')$\to$dom(ξ) st. $\xi' \sqsubseteq \xi \circ F$

Main Theorem [KW'85]: Fix admissible $\xi :\to X$ and $\upsilon :\to Y$.
$f: X \to Y$ is continuous **iff** it has a continuous (ξ,υ)-realizer.

Standard Representation

"Kolmogorov"

Def: Fix a topological T_0 space X with subbasis O_n, $n\in\mathbb{N}_+$.
The standard representation ξ of X (wrt. O_n) is the following:
A ξ–name of $x\in X$ is a list of all $n\in\mathbb{N}$ (in any order) with $x\in O_n$.

Theorem: The standard representation <u>is</u> admissible.

Proof: (i) $\xi^{-1}[O_n\cap O_m] = \bigcup_{k,\ell} \{ \underline{u}\in \mathcal{B}: u_k=n \wedge u_\ell=m \}$ open √

(ii) $\xi'\preceq\xi$ for every continuous (not necessarily surj.) $\xi':\to X$

Let $F(\underline{v})_{\langle m,j\rangle} := m$ if $\xi'[(v_0,v_1 \ldots v_j) \circ \mathbb{Z}^{\mathbb{N}}] \subseteq O_m$, $:= 0$ else.

Def: Representation ξ of X is admissible if (i) is continuous and (ii) every continuous representation ξ' of X has: $\xi'\preceq\xi$.

Reduction $\xi'\preceq\xi$ means cont. F:dom(ξ')→dom(ξ) st. $\xi' \sqsubseteq \xi\circ F$

Representation of X is a surjective partial mapping $\xi:\to X$.

$\mathbb{N}$–Pairing bijection "*Hilbert Hotel*" $\langle x,y\rangle = x + (x+y)\cdot(x+y+1)/2$

Cartesian Product Representation

Def a) For representations ξ of X and υ of Y, write $\xi\times\upsilon$ for the representation of $X\times Y$ with $(\xi\times\upsilon)(\langle \underline{u},\underline{v}\rangle) := (\xi(\underline{u}), \upsilon(\underline{v}))$.

b) For representations ξ_j of X_j, $j\in J\subseteq\mathbb{N}$, write $\prod_j \xi_j$ for the representation of $\prod_{j\in J} X_j$ with $\prod_j \xi_j(\langle \underline{u}_0,\underline{u}_1,\underline{u}_2,\ldots\rangle) := (\xi_j(\underline{u}_j))_j$

Examples:

Recall $\delta{:\to}\mathbb{R}$

$\delta^{\mathbb{N}}{:\to}\mathbb{R}^{\mathbb{N}}$ sequences of reals

$\delta^{*}{:\to}\mathbb{R}^{*}$ vectors of reals

$\delta^{*}{:\to}\mathbb{R}[X]$ real polynomials $\subseteq\mathbb{C}^{*}$

$\delta^{*\mathbb{N}}{:\to}(\mathbb{R}[X])^{\mathbb{N}}$ sequences of polynomials

$\mathcal{B}/\mathcal{C}$–<u>binary</u> pairing

$\mathcal{B}/\mathcal{C}$–<u>countable</u> pairing $\langle \underline{u}_0,\underline{u}_1,\underline{u}_2,\ldots\rangle_{\langle j,n\rangle} := u_{j,n}$

Lemma: If $\xi{:\to}X$ and $\upsilon{:\to}Y$ and $\xi_j{:\to}X_j$ are admissible, then so are $\xi\times\upsilon{:\to}X\times Y$ and $\prod_{j\in J}\xi_j \to \prod_{j\in J} X_j$.

Representing Functions & Sets

Recall: To compute $\Xi :\subseteq C(K) \to C(K')$ means:

Convert *any* $(P_m) \subseteq \mathbb{D}[X]$ with $\|f - P_m\|_\infty \leq 2^{-m}$

to some $(Q_n) \subseteq \mathbb{D}[X]$ with $\|g - Q_n\|_\infty \leq 2^{-n}$, $g = \Xi(f)$.

$\delta^{\mathbb{N}} :\to \mathbb{R}^{\mathbb{N}}$ sequences of reals

$\delta^{*} :\to \mathbb{R}^{*}$ vectors of reals

$\delta^{*} :\to \mathbb{R}[X]$ real polynomials $\subseteq \mathbb{C}^{*}$

$\delta^{*\mathbb{N}} :\to (\mathbb{R}[X])^{\mathbb{N}}$ sequences of polynomials

$\delta_{\square} :\to C(K)$ continuous functions

$\kappa :\to \mathcal{K}(X)$ compact subsets

$d_A : X \ni \underline{x} \to \inf\{\, d(\underline{x},\underline{a}) : \underline{a} \in A \,\}$

Representation of X is a surjective partial mapping $\xi :\to X$.

$\mathbb{N}$–Pairing bijection "*Hilbert Hotel*" $\langle x,y \rangle = x + (x+y)\cdot(x+y+1)/2$

Representing Functions & Sets II

KAIST

CS700 M. Ziegler

Recall: To compute $f{:}K\subseteq\mathbb{R}\to\mathbb{R}$ also means:
Compute real 'sequence' $f(q)$, $q\in\mathbb{D}\cap K$ and
compute a modulus $\mu{:}\mathbb{N}\to\mathbb{N}$ of continuity of f.

$2_A(\underline{x},n) = +$ if $d_A(\underline{x})\leq 2^{-n}$,

$2_A(\underline{x},n) = -$ if $d_A(\underline{x})\geq 2^{-n-1}$

$\underline{x}\in\mathbb{D}$

$\kappa'{:}\to\mathcal{K}(X)$

$\kappa{:}\to\mathcal{K}(X)$

compact subsets

$d_A{:}X\ni\underline{x}\to\inf\{\ d(\underline{x},\underline{a}) : \underline{a}\in A\ \}$

$\delta^{\mathbb{N}}{:}\to\mathbb{R}^{\mathbb{N}}$ sequences of reals

$\delta^{*}{:}\to\mathbb{R}^{*}$ vectors of reals

$\delta^{*}{:}\to\mathbb{R}[X]$ real polynomials

$\delta_{\blacksquare}{:}\to C(K)$ continuous functions

$\delta_{\square}{:}\to C(K)$ continuous functions

Reduction $\xi'\preceq\xi$ means cont. $F{:}\mathrm{dom}(\xi')\to\mathrm{dom}(\xi)$ st. $\xi' = \xi\circ F$

Theorem: $\delta_{\square}\equiv\delta_{\blacksquare}$ are admissible, $\kappa\equiv\kappa'$ are admissible.

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