IV. Recap of Discrete Complexity

- Computation with Cost
- Complexity of Arithmetic
- Selected Complexity Classes
- Classifying Example Decision Problems
- Comparing Decision Problems
- Complexity Class Picture

IV. Recap of Discrete Complexity (2)

- Complexity Class Picture
- *Counting* Complexity Class $\#\mathcal{P}$
- *Unary* Complexity Classes
- *Parameterized* Complexity
- *Average*-Case Complexity

Computation with Cost

$\mathcal{P} := (x_j := 0 \mid x_j := 1 \mid x_j := x_i + x_k \mid x_j := x_i \ominus x_k$
$\mid x_j := x_i \oslash 2 \mid \mathcal{P} ; \mathcal{P} \mid \text{WHILE } x_j \text{ DO } P \text{ END})$

n	$\log_2 n \cdot 10$s	$n \cdot \log n$ sec	n^2 msec	n^3 µsec	2^n nsec
10	33sec	33sec	0.1sec	1msec	1msec
100	≈1min	11min	10sec	1sec	40 Mrd. Y
1000	≈1.5min	≈3h	17min	17min	
10 000	≈2min	1.5 days	≈1 day	11 days	
100 000	≈2.5min	19 days	4 months	32 years	

Definitions: binary *length* of $x \in \mathbb{N}$: $\ell(x) = \lceil \log_2(1+x) \rceil$

- time of a WHILE+ program P on input $\underline{x}=(x_1,\ldots x_k)$
- space (=memory) used: $\max \{\ell(x_1),\ldots \ell(x_k)\}$
- asymptotic time/space $t(n)/s(n)$:
  worst-case over all inputs $\underline{x}$ with $\ell(\underline{x})<n$

Complexity of Arithmetic

$\mathcal{P} := (\ x_j := 0 \mid x_j := 1 \mid x_j := x_i + x_k \mid x_j := x_i \ominus x_k$
$\mid x_j := x_i \oslash 2 \mid \mathcal{P} ; \mathcal{P} \mid \text{WHILE } x_j \text{ DO } P \text{ END }\ )$

- Multiplication by repeated addition: expon. time
- Long multiplication: linear time
- Long division, un-/pairing: polyn. time

$\mathbb{N}$–Pairing $\langle x,y\rangle = x+(x+y)\cdot(x+y+1)/2$: $\ell(\langle x,y\rangle) \leq O(\ \ell(x)+\ell(y))$

Definitions: binary *length* of $x\in\mathbb{N}$: $\ell(x) = \lceil \log_2(1+x) \rceil$

- time of a WHILE+ program P on input $\underline{x}=(x_1,\ldots x_k)$
- space (=memory) used: $\max\{\ell(x_1),\ldots\ell(x_k)\}$

Lemma: In each step, $X:=\max\{x_1,\ldots x_k\}$ at most doubles, and $\max\{\ell(x_1),\ldots\ell(x_k)\}$ increases by ≤ 1.

Selected Complexity Classes

Proposition:

a) If f,g are polyn-time computable, then so is $g\circ f$.

b) If a WHILE+ program using $\leq s(x)$ memory makes $\geq\Omega(2^{s(x)})$ steps, it does not terminate.

Def: For decision problems $L\subseteq\mathbb{N}$ or $L\subseteq\{0,1\}^*$

- $\mathcal{P}$ = { L decidable in polynomial time }
- $\mathcal{NP}$ = { L *verifiable* in polynomial time }, i.e.

$$L = \{\, x\in\mathbb{N} : \exists y\in\mathbb{N},\ \ell(y)\leq\mathrm{poly}(\ell(x)),\ \langle x,y\rangle\in V \,\},\ \ V\in\mathcal{P}$$

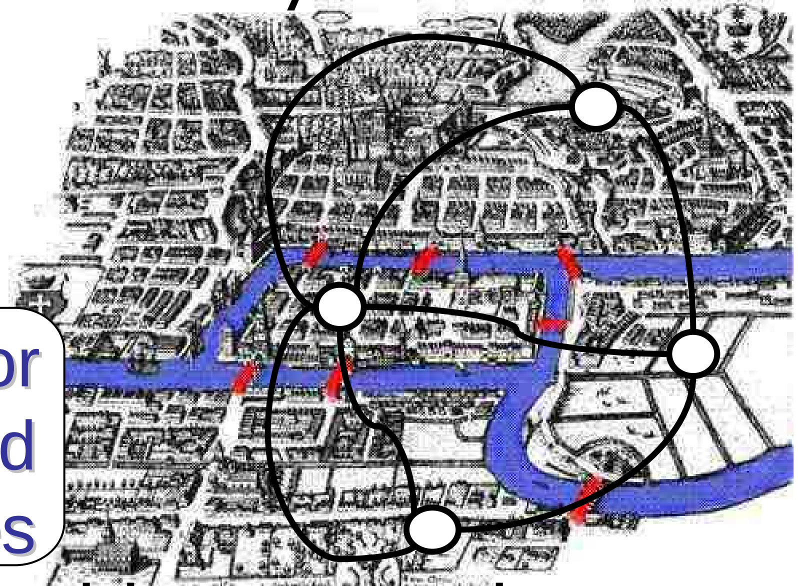
- $\mathcal{PSPACE}$ = { L decidable in polynomial space }
- $\mathcal{EXP}$ = { L decidable in exponential time }

Theorem: $\mathcal{P}\subseteq\mathcal{NP}\subseteq\mathcal{PSPACE}\subseteq\mathcal{EXP}$

Example Decision Problems

In an undirected graph *G*, Eulerian cycle traverses each <u>edge</u> precisely once;

Hamiltonian cycle visits each <u>vertex</u> precisely once.



G admitting a Eulerian cycle is connected ~~and~~ has an even number of edges incident to each vertex

save for isolated vertices

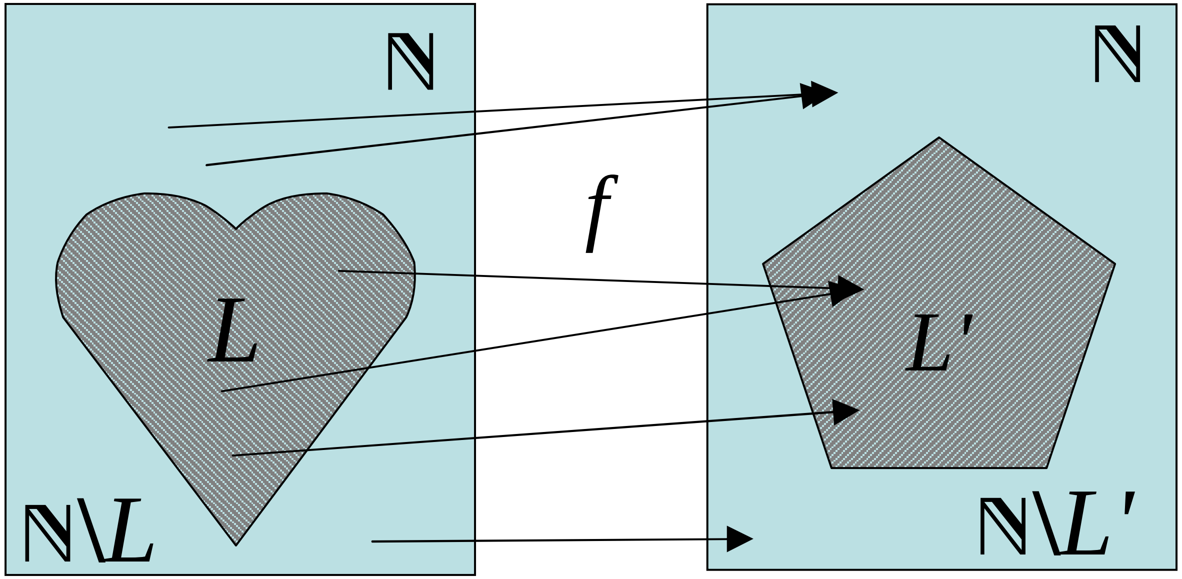
Theorem: Conversely every connected graph with an even number of edges incident to each vertex admits a Eulerian cycle.

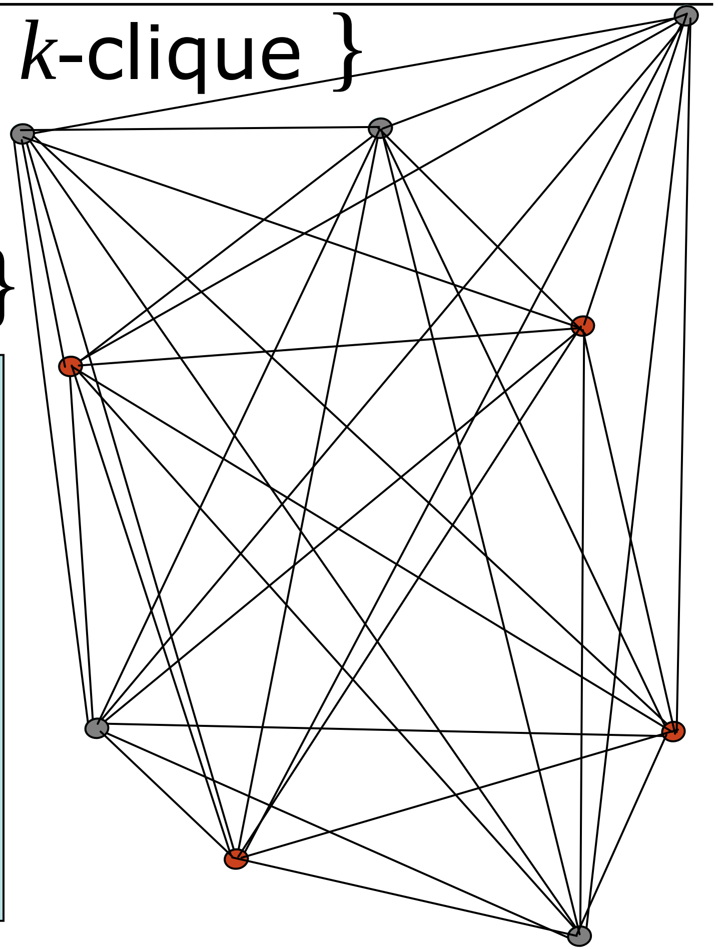
EC := { ⟨*G*⟩ | *G* has a Eulerian cycle}	$\mathcal{NP}$
HC := { ⟨*G*⟩ | *G* has Hamiltonian cycle}	$\mathcal{NP}$

Comparing Decision Problems 2

CLIQUE = { ⟨*G*,*k*⟩ | *G* contains a *k*-clique }

$\equiv_p$ **IS**={ ⟨*G*,*k*⟩ : *G* has *k* pairwise non-connected vertices}





For $L,L'\subseteq\mathbb{N}$ write $L\preccurlyeq_p L'$ if exists a polynomial-time computable $f:\mathbb{N}\to\mathbb{N}$ such that $\forall \underline{x}$: $\underline{x}\in L \Leftrightarrow f(\underline{x})\in L'$

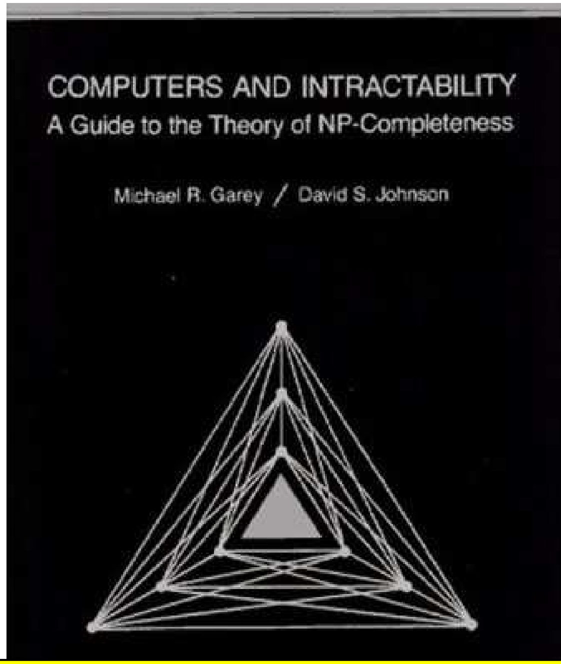
Lemma: a) $L\preccurlyeq_p L'\preccurlyeq_p L'' \Rightarrow L\preccurlyeq_p L''$ b) $L'\in\mathcal{P}\Rightarrow L\in\mathcal{P}$

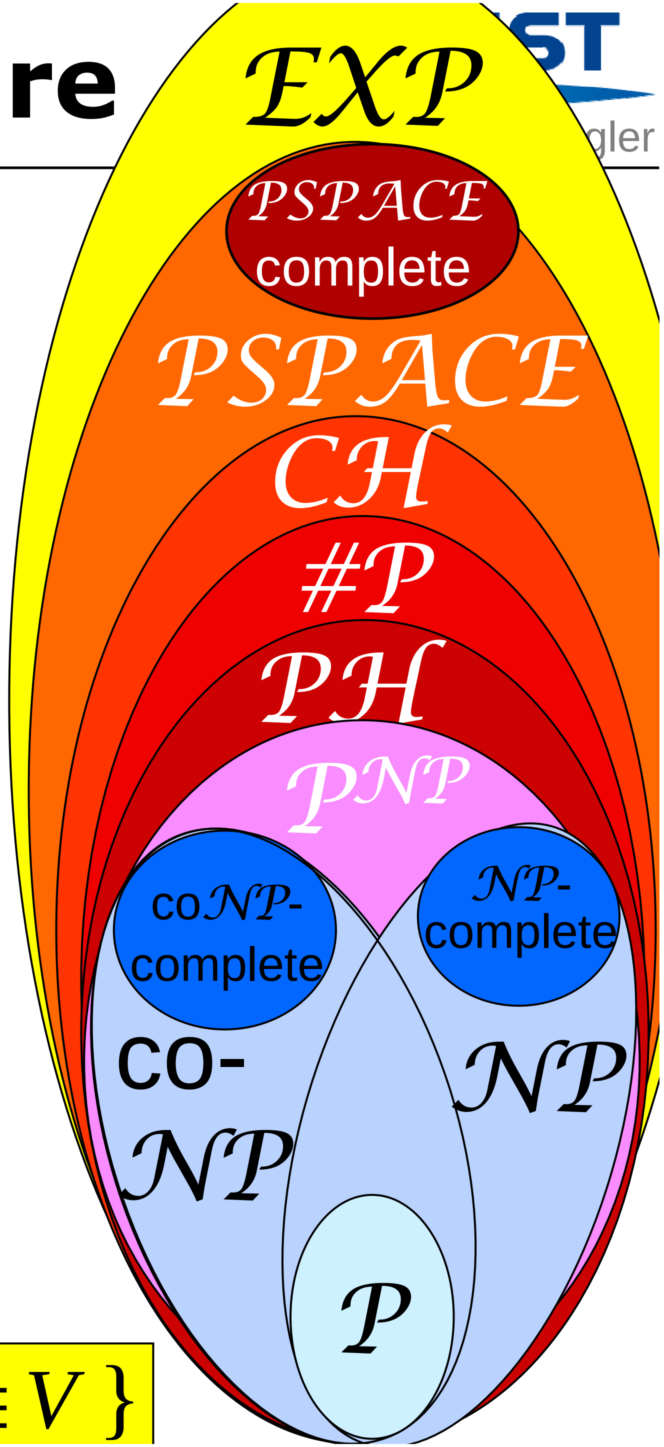
Complexity Class Picture

Def: $A \in \mathcal{NP}$ is $\mathcal{NP}$-complete if $L \preccurlyeq_p A$ holds for every $L \in \mathcal{NP}$.

Theorem (Cook'72/Levin'71):
SAT is $\mathcal{NP}$-complete!

There exists a $V \in \mathcal{P}$ such that the following is $\mathcal{NP}$-complete:





$\{\ x \in \mathbb{N} : \exists y \in \mathbb{N},\ \ell(y) \leq \ell(x),\ \ \langle x,y \rangle \in V\ \}$

Counting Complexity Class $\#\mathcal{P}$

- $\mathcal{P} = \{\, L \subseteq \mathbb{N}$ decidable in polynomial time $\}$
- $\mathcal{NP} = \{\, L \subseteq \mathbb{N}$ *verifiable* in polynomial time $\}$,

$$L = \{\, x \in \mathbb{N} \;:\; \exists y \in \mathbb{N},\ \ell(y) \leq \mathrm{poly}(\ell(x)),\ \langle x,y \rangle \in V \,\},\quad V \in \mathcal{P}$$

- $\#\mathcal{P} = \{\, f : \mathbb{N} \to \mathbb{N}$ *countable* in polynomial time $\}$

$$f(x) = \#\{\, y \in \mathbb{N} \;:\; \ell(y) \leq \mathrm{poly}(\ell(x)),\ \langle x,y \rangle \in V \,\},\quad V \in \mathcal{P}$$

- $\mathcal{PSPACE} = \{\, L \subseteq \mathbb{N}$ decidable in polyn. space $\}$
- $\mathcal{EXP} = \{\, L \subseteq \mathbb{N}$ decidable in exponential time $\}$

Theorem: $\mathcal{P} \subseteq \mathcal{NP} \subseteq \#\mathcal{P} \subseteq \mathcal{PSPACE} \subseteq \mathcal{EXP}$

Unary Complexity Classes

Definitions: binary *length* of $x\in\mathbb{N}$: $\ell(2^n) = n+1$

Time/space $t(n)$/s(n) worst-case over inputs $x\in\mathbf{2}^{\mathbb{N}}$ $\ell(\underline{x})<n$

- $\mathcal{P}_1 = \{\ L\subseteq\mathbf{2}^{\mathbb{N}}$ decidable in polyn. time $\}$ $\{\ 2^n : n\in\mathbb{N}\ \}$ $=: \mathbf{2}^{\mathbb{N}}$
- $\mathcal{NP}_1 = \{\ L\subseteq\mathbf{2}^{\mathbb{N}}$ *verifiable* in polyn. time $\}$,

$$L = \{\ x\in\mathbf{2}^{\mathbb{N}} : \exists y\in\mathbb{N},\ \ell(y)\leq\mathrm{poly}(\ell(x)),\ \langle x,y\rangle\in V\ \},\quad V\in\mathcal{P}$$

- $\#\mathcal{P}_1 = \{\ f:\mathbf{2}^{\mathbb{N}}\to\mathbb{N}$ *countable* in polynomial time $\}$

$$f(x) = \#\{\ y\in\mathbb{N} : \ell(y)\leq\mathrm{poly}(\ell(x)),\ \langle x,y\rangle\in V\ \},\quad V\in\mathcal{P}$$

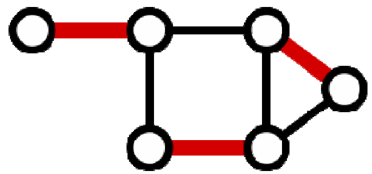
- $\mathcal{PSPACE}_1 = \{\ L\subseteq\mathbf{2}^{\mathbb{N}}$ decidable in polyn. space $\}$
- $\mathcal{EXP}_1 = \{\ L\subseteq\mathbf{2}^{\mathbb{N}}$ decidable in exponential time $\}$

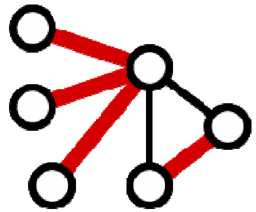
Theorem: $\mathcal{P}_1 \subseteq \mathcal{NP}_1 \subseteq \#\mathcal{P}_1 \subseteq \mathcal{PSPACE}_1 \subseteq \mathcal{EXP}_1$

Parameterized Complexity

- **Edge Cover (EC)** $\mathcal{P}$ — Greedily extend a maximum matching

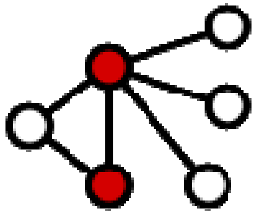
"To graph G, find a <u>smallest</u> subset F of edges s.t. any vertex v is adjacent to at least one $e \in F$."

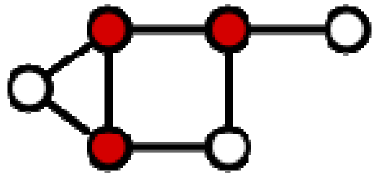




- vs. **Vertex Cover (VC)**

$\mathcal{NPc}$





There is an algorithm deciding **EC** in time $\text{poly}(n)\cdot O(1)^k$: "Efficient in n for 'small' k" — class $\mathcal{FPT}$

Call $n\in\mathbb{N}$ *primary* cost parameter, $k\in\mathbb{N}$ *secondary*.

$$\mathbf{VC} = \{\langle V,E,k\rangle : \exists U\subseteq V,\ |U|=k,\ \forall (x,y)\in E:\ x\in U \vee y\in U\ \}$$

$$\mathbf{EC} = \{\langle V,E,k\rangle : \exists F\subseteq E,\ |F|=k,\ \forall x\in V\ \exists y\in V:\ (x,y)\in F\ \}$$

***Average*-Case Complexity**

QuickSort (e.g. with first/last element as pivot)

has *quadratic* runtime: as bad as **BubbleSort**

— in the *worst*-case, e.g. on *sorted* inputs.

rare

Theorem: For a **random** input, picked uniformly among all $N!$ possible inputs of size N, the number $C=C(N)$ of comparisons performed by **QuickSort** has **expected** value $\leq 2N \cdot \ln(N)$.

random variable

probability distribution

Def [Levin'86]: class $\mathit{dist}\mathcal{NP}$ of *average*-case $\mathcal{NP}$ problems $L \subseteq \mathbb{N}$ wrt *given* probability distribution D.

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