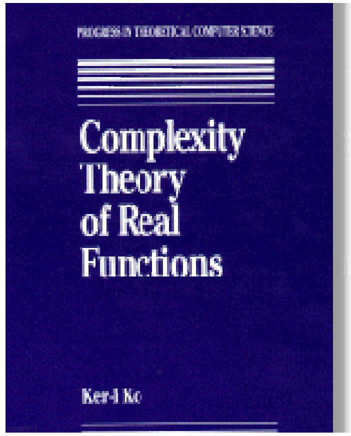
V. Complexity Theory over the Reals



- *Polytime*-computable real numbers
- *Polytime*-computable real functions
- *Quantitative* computability ⇔ quantitative continuity
- Operations that preserve *polytime* functions
- (Strongly) *polytime*-computable real sequences
- Polytime analytic functions ⇔ polytime Taylor series
- Operations that preserve *polytime* <u>analytic</u> functions

V. Complexity Theory over Reals (2)

- *Parametric* Maximization "*in* $\mathcal{NP}$"
- *Indefinite* Riemann Integration "*in* $\#\mathcal{P}$"
- *Definite* Riemann Integration "*in* $\#\mathcal{P}_1$"
- ODESOLVE "*in* $\mathcal{PSPACE}$"
- Parametric Maximization is $\mathcal{NP}$-"*complete*"
- In/definite integration is $\#\mathcal{P}/\#\mathcal{P}_1$-"*complete*"
- Complexity of PDEs: Poisson and Heat Equation
- More numerical characterizations of discrete complexity classes

Complexity of Real Numbers

Problem: Determine first digit of **`0.111…1`** $\pm 2^{-n}$ **?**

$r<0 \;\Leftrightarrow\; \exists n: a_n<1$ and $r>0 \;\Leftrightarrow\; \exists n: a_n>1$

Theorem: For $r\in\mathbb{R}$, the following are equivalent:

a) r has a decidable binary expansion
$\{ n : b_n=1 \} \subseteq \mathbb{N}$ for $r = \sum_n b_n/2^n$.

b) There exists an algorithm computing a sequence $(a_n)\subseteq\mathbb{Z}$ with $|r-a_n/2^n| \leq 2^{-n}$

c) There exist algorithms computing sequences $(a_m),(b_m),(c_m)\subseteq\mathbb{Z}$ with $|r-a_m/b_m| \leq 1/c_m \to 0$

$\Uparrow\!\!\!\not\;\Downarrow$ **poly-time** $\Uparrow\!\!\!\not\;\Downarrow$

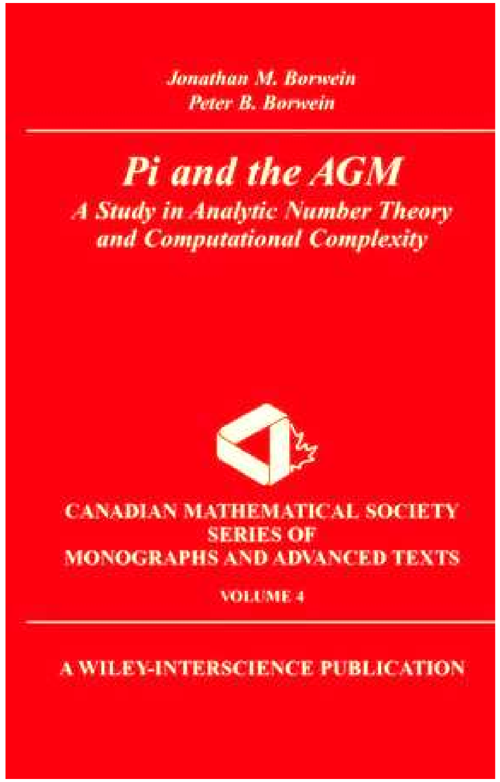
Def: Computing $r\in\mathbb{R}$ in time $t:\mathbb{N}\to\mathbb{N}$ means to output $a_0,\ldots a_n$ in $\leq t(n)$ steps.

$\text{poly}(n) := O(n)^{O(1)}$

Polytime-Computable Reals

Example: The following are polytime computable:

- sum, product, and reciproke of any polytime-computable real(s)
- every single algebraic real
- some transcendental reals such as e=2.718.. or π.



sequence $(a_n)\subseteq\mathbb{Z}$ with $|r-a_n/2^n|\leq 2^{-n}$

Example:
Not polytime computable is $\sum_{j\in L}4^{-j}$ for $\mathbb{N}\supseteq L\notin\mathcal{P}_1$

Def: Computing $r\in\mathbb{R}$ in time $t:\mathbb{N}\to\mathbb{N}$ means to output $a_0,\ldots a_n$ in $\leq t(n)$ steps.	$\mathrm{poly}(n) := O(n)^{O(1)}$

Proof (Sketch)

Example: The following are polytime computable:

Let $p\in\mathbb{Z}[X]$ be minimal polynomial to $r\in\mathbb{A}$.
Use bisection to approximate simple root r of p.

- every single algebraic real

$\sum_{j\in L}4^{-j}$ polytime iff $4/3+\sum_{j\in L}4^{-j} = \sum_{j\in\mathbb{N}} \begin{cases} \mathbf{2}\cdot 4^{-j} : j\in L \\ \mathbf{1}\cdot 4^{-j} : j\notin L \end{cases}$

sequence $(a_n)\subseteq\mathbb{Z}$ with $|r-a_n/2^n|\leq 2^{-n}$

Example:

Not polytime computable is $\sum_{j\in L}4^{-j}$ for $\mathbb{N}\supseteq L\notin \mathcal{P}_1$

$\sum_{j\in L}2^{-j}$ is binary decidable in polytime iff $\mathbb{N}\supseteq L\in \mathcal{P}_1$

First digit of **0.111…1** $\pm 2^{-n}$ **?** **true=10**, **false=01**

Complexity of Continuous Functions

Fact (*Bernstein's Theorem*)**:** To approximate $|x-½|$ up to 2^{-n} requires polynomials of degree expon. in n

b) There is an algorithm printing a sequence (of coefficient lists of) $(P_n)\subseteq\mathbb{D}[X]$ with $\|f-P_n\|_\infty \leq 2^{-n}$

poly-time ⇑̸ ⇓

a) There is an algorithm converting any $\underline{a}=(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m| \leq 2^{-m}$, to $(b_n)\in\mathbb{Z}$ with $|f(x)-b_n/2^n| \leq 2^{-n}$

⇕

c) 'Sequence' $f(q)\in\mathbb{R}$, $q\in\mathbb{D}\cap[0,1]$, is *strong* polytime

∧ f admits a polynomial modulus of continuity

$$|x-y|\leq 2^{-\mu(n)} \Rightarrow |f(x)-f(y)|\leq 2^{-n}$$

poly-time

Def: Computing $f:\subseteq\mathbb{R}\to\mathbb{R}$ in time $t:\mathbb{N}\to\mathbb{N}$ means to output $b_0,\ldots b_n$ in $\leq t(n)$ steps **regardless of** x, (a_m).

Computability ⇒ Qualit.Continuity Complexity ⇒ Quantit. Continuity

Recall compute $F:\subseteq\mathcal{B}\to\mathcal{B}$: On input $\underline{u}\in\mathrm{dom}(F)$ output $F(\underline{u})$.

Compute F in time $t:\mathbb{N}\to\mathbb{N}$: $F(u)_n$ appears after $\le t(n)$ steps.

$\mathcal{B} = \mathbb{Z}^{\mathbb{N}}$ — **regardless** of $\underline{u}$

Basic *Main* Theorem:

b) F computable in time t ⇒ t is a modulus of continuity of F.

d) $\mathrm{dom}(F)$ compact, $\mathcal{A}$ computes F ⇒ has time bound $t=t(n)$

Real Theorem b') If $f:\subseteq\mathbb{R}\to\mathbb{R}$ is *polytime* computable, then f has a *polynomial* modulus of continuity.

Real Theorem d'): If $f:[0;1]\to\mathbb{R}$ is computable, then so within bounded time $t(n)$ for some $t:\mathbb{N}\to\mathbb{N}$.

Example: $\exp:\mathbb{R}\to\mathbb{R}$ is <u>not</u> computable in bounded time, but on $[-2^k;k]$ computable in time $\mathrm{poly}(n+k)$.

Proof of the <u>Real</u> *Main Theorem*

Theorem: δ *proper*: compact $K\subseteq\mathbb{R} \Rightarrow \delta^{-1}[K]\subseteq\mathcal{B}$ compact

2) $f:\subseteq\mathbb{R}\to\mathbb{R}$ has a *polynomial* modulus of continuity iff has a (δ,δ)-realizer $F:\subseteq\mathcal{B}\to\mathcal{B}$ with *polynom*. modulus.

b) F computable in time $t \Rightarrow t$ is a modulus of continuity of F.

d) $\mathrm{dom}(F)$ compact, $\mathcal{A}$ computes $F \Rightarrow$ has time bound $t=t(n)$

Real Theorem b') If $f:\subseteq\mathbb{R}\to\mathbb{R}$ is *polytime* computable, then f has a *polynomial* modulus of continuity.

Real Theorem d'): If $f:[0;1]\to\mathbb{R}$ is computable, then so within bounded time $t(n)$ for some $t:\mathbb{N}\to\mathbb{N}$.

dyadic representation: $|r- a_n/2^n| \leq 2^{-n}$ $\quad\delta:\subseteq\mathcal{B}\to[0;1]$

A (ξ,υ)-realizer of $f:X\to Y$ is a $F:\mathrm{dom}(\xi)\to\mathrm{dom}(\upsilon)$ s.t. $f\circ\xi \sqsubseteq \upsilon\circ F$.

Operations on Polytime Functions

Theorem: Let $f:[-1;0]\to\mathbb{R}$, $g:[0;1]\to\mathbb{R}$ *polytime* with $f(0)=g(0)$. Then $f\cup g:[-1;1]\to\mathbb{R}$ is *polytime*.

Theorem: Suppose *differentiable* $f:[0;1]\to\mathbb{R}$ is *polytime* and f' has a *polynomial* modulus of continuity. Then f' is again *polytime* computable.

Theorem: Let $f:[0;1]\to[0;1]$ bijective & *polytime* s.t. its inverse f^{-1} has a polynomial modulus.
Then its inverse $f^{-1}:[0;1]\to[0;1]$ is *polytime*, too.

'Sequence' $f(q)\in\mathbb{R}$, $q\in\mathbb{D}\cap[0,1]$, is *strongly polytime*
$\wedge$ f admits a *polynomial* modulus of continuity

(Strongly) Polytime Real Sequences

Def: Computing sequence $(r_j) \subseteq \mathbb{R}$ in **polytime**
means to output $(a_n) \subseteq \mathbb{Z}$ with $|r_j - a_{\langle j,m\rangle}/2^m| \leq 2^{-m}$
s.t. $a_0, \ldots a_n$ appears in $\mathrm{poly}(n) = \mathrm{poly}(m+j)$ steps

$\langle x,y\rangle = x + (x+y)\cdot(x+y+1)/2,$ $\quad j+m \leq n = \langle j,m\rangle \leq (j+m)^2$

n, j+m = polyn. length of *unary* encoding

m in *unary*, *j* in *binary*

Def: Strongly computing $(r_j) \subseteq \mathbb{R}$ in polytime
means to output $(a_n) \subseteq \mathbb{Z}$ with $|r_j - a_{\langle j,m\rangle}/2^m| \leq 2^{-m}$
s.t. $a_0, \ldots a_n$ appears in $\mathrm{poly}(m+\log j)$ steps.

Def: Computing $r \in \mathbb{R}$ in time $t: \mathbb{N} \to \mathbb{N}$ means to output $a_0, \ldots a_n$ in $\leq t(n)$ steps.	$\mathrm{poly}(n) := O(n)^{O(1)}$

Polytime Analytic Functions

Def: Computing sequence $(r_j)\subseteq\mathbb{R}$ in **polytime**
means to output $(a_n)\subseteq\mathbb{Z}$ with $|r_j-a_{\langle j,m\rangle}/2^m|\leq 2^{-m}$
s.t. $a_0,\ldots a_n$ appears in $\mathrm{poly}(n)$ = $\mathrm{poly}(m+j)$ steps

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Theorem: Fix $(c_j)\subseteq\mathbb{R}$ and $r<R=1/\limsup_j |c_j|^{1/j}$.
(c_j) is polytime **iff** $[-r;r]\ni x\to\sum_j c_j\cdot x^j$ is polytime.

Def: Computing $f:\subseteq\mathbb{R}\to\mathbb{R}$ in time $t:\mathbb{N}\to\mathbb{N}$ means,
on input of $(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^m|\leq 2^{-m}$, regardless
to output b_n within $\leq t(n)$ steps of $x\in\mathrm{dom}(f)$, (a_m).

Def: Computing $r\in\mathbb{R}$ in time $t:\mathbb{N}\to\mathbb{N}$
means to output $a_0,\ldots a_n$ in $\leq t(n)$ steps.

$\mathrm{poly}(n) := O(n)^{O(1)}$

Proof (Sketch)

Lagrange Interpolation, [Ko91, p].206ff]

Recall computing $f^{(j)}(x)$**:** Given $n\in\mathbb{N}$ output $(f(x+\delta)-f(x))/\delta$.

$f'(y) = (f(x+\delta)-f(x))/\boldsymbol{\delta}$ for <u>some</u> $y\in[x,x+\delta]$: $\delta:=2^{-n}/\|f''\|_\infty$

by the *Mean Value Theorem.* Then $|\, f'(y)- f'(x)\,| \leq 2^{-n}$.

Theorem: Fix $(c_j)\subseteq\mathbb{R}$ and $r<R=1/\limsup_j |c_j|^{1/j}$.

(c_j) is polytime **iff** $[-r;r] \ni x \to \sum_j c_j\cdot x^j$ is polytime.

Proof "$\Rightarrow$": Since $r<s<R$, $\exists B\in\mathbb{N}$ $\forall j$: $|c_j| \leq B/s^j$.

Tail bound $|\sum_{j>J} c_j\cdot z^j| \leq B\cdot(r/s)^J \cdot$ geometr.series $\leq 2^{-n}$

Fact: f analytic on $[-r;r]$

$\Rightarrow$ $|f^{(j)}(x)| \leq j! \cdot C \cdot (r/2)^{-j}$ on $[-r/2;r/2]$ for some $C\in\mathbb{N}$

Operations on Analytic Functions

$\mathrm{Max}(f): x \to \max\{\, f(t): t \le x \,\}$

Fact: Maximizing polynomials is polytime computable.

Theorem: Fix $(c_j) \subseteq \mathbb{R}$ and $r < R = 1/\limsup_j |c_j|^{1/j}$.
(c_j) is polytime **iff** $[-r;r] \ni x \to \sum_j c_j \cdot x^j$ is polytime.

Corollary: Fix analytic $f:(-R;R)$, polytime on $[-r;r]$.

a) f' and $f^{(j)}$ are again polytime on $[-r;r]$

b) $\int f$ is again polytime on $[-r;r]$

c) $\mathrm{Max}(f)$ is again polytime on $[-r;r]$

Complexity of Parametric Maximization

Fix polytime f:[0;1]→[0;1] with modulus $\mu\in\mathbb{N}[n]$.

$\mathrm{Max}(f): x \to \max\{ f(t): t \le x \}$

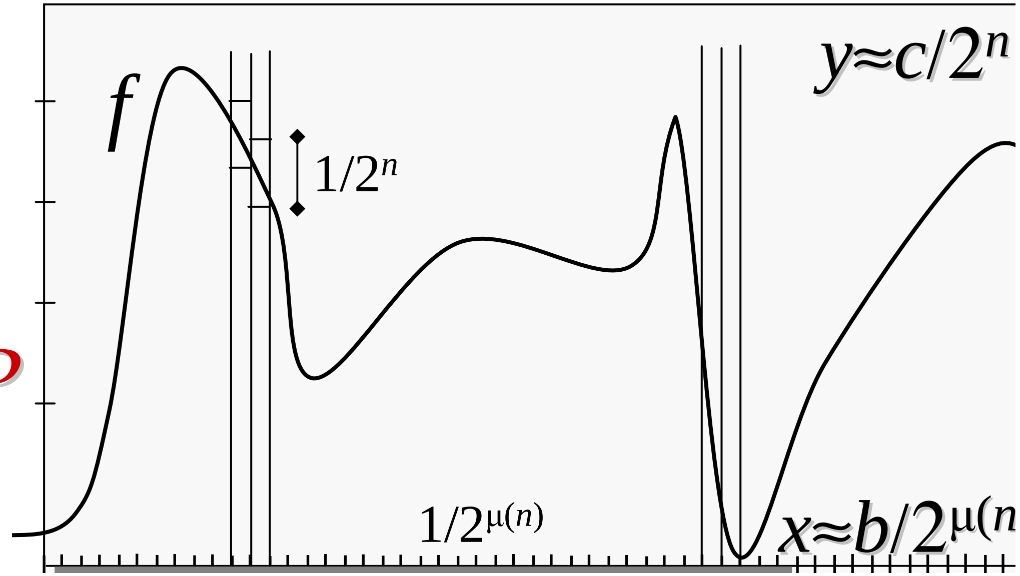
- is computable in $\mathcal{EXP}$ / $\mathcal{PSPACE}$
- is polytime, provided that $\mathcal{P}=\mathcal{NP}$:

Let $\varphi_n :\subseteq \mathbb{N}\to\mathbb{N}$ be computable in time polynom. in n s.t. $| f(a/2^{\mu(n)}) - \varphi_n(a)/2^n | \le 2^{-n}$ for all $a\in\{0,\ldots 2^{\mu(n)}-1\}$

and employ

$\{ (2^n,b,c) \mid b < 2^{\mu(n)},\ c < 2^n,\ \exists a\le b: \varphi_n(a) \ge c \} \in \mathcal{NP}$

for bisection w.r.t. c.



Complexity of Indefinite *Integration*

Fix polytime f:[0;1]→[0;1] with modulus $\mu\in\mathbb{N}[n]$.

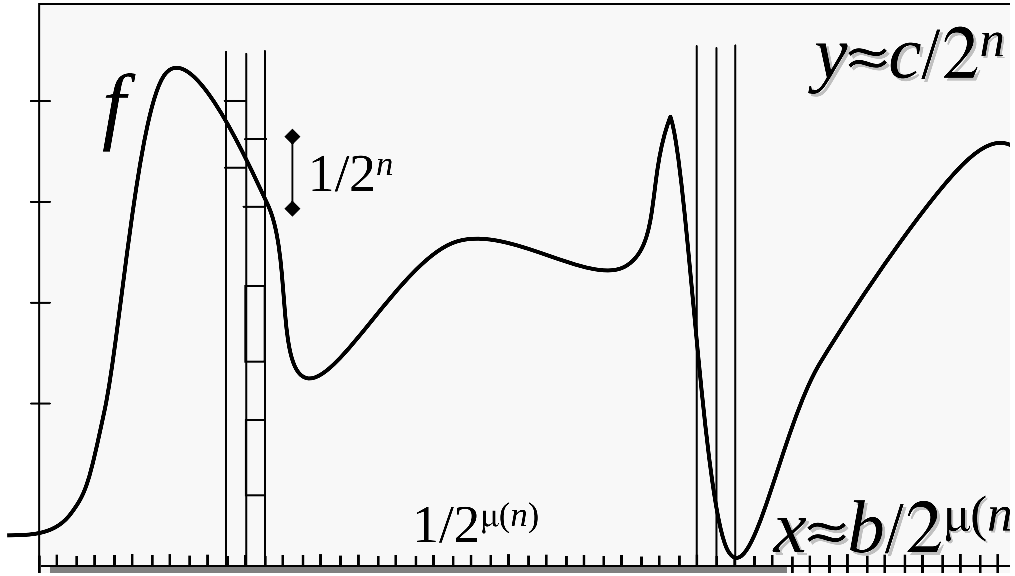
$$\int f : x \to \int^x f(t)\,dt$$

- is computable in $\mathcal{EXP}$ / $\mathcal{PSPACE}$
- is polytime, provided that $\mathcal{FP}=\#\mathcal{P}$:

Let $\varphi_n :\subseteq \mathbb{N}\to\mathbb{N}$ be computable in time polynom. in n s.t. $|\, f(a/2^{\mu(n)}) - \varphi_n(a)/2^n \,| \le 2^{-n}$ for all $a\in\{0,\ldots 2^{\mu(n)}-1\}$

and employ

$$(2^n,b) \to \#\{\ (a,c)\ |\ c < 2^n ,\ a \le b\ ,\ \varphi_n(a) \ge c\ \} \in \#\mathcal{P}$$



Complexity of *De*finite *Integration*

Fix polytime $f:[0;1]\to[0;1]$ with modulus $\mu\in\mathbb{N}[n]$.

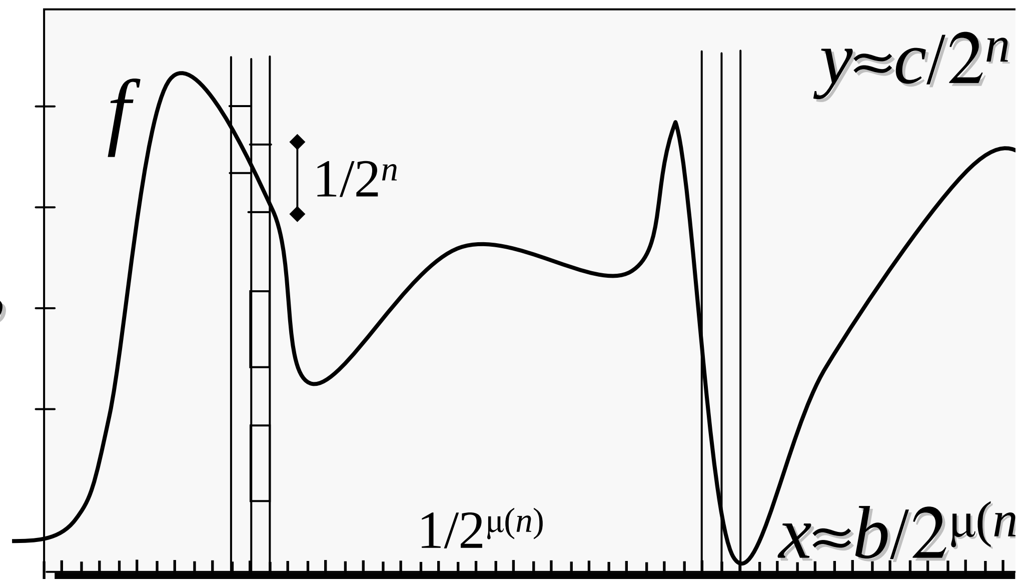
$$\int^{1} f(t)\,dt$$

- is computable in $\mathcal{EXP}_1$ / $\mathcal{PSPACE}_1$
- is polytime, provided that $\mathcal{FP}_1=\#\mathcal{P}_1$:

Let $\varphi_n:\subseteq\mathbb{N}\to\mathbb{N}$ be computable in time polynom. in n s.t. $|\,f(a/2^{\mu(n)}) - \varphi_n(a)/2^n\,| \leq 2^{-n}$ for all $a\in\{0,\ldots 2^{\mu(n)}-1\}$

and employ

$$2^n \to \#\{\,(a,c)\mid c<2^n,\ a\leq 2^{\mu(n)},\ \varphi_n(a)\geq c\,\}\in\#\mathcal{P}_1$$



Complexity of ODESOLVE

Fix polytime f:[0;1]×[-1;1]→[-1;1]

generalizes indefinite integration $\dot{z}(t)=f(t,z),\ \ z(0)=0$

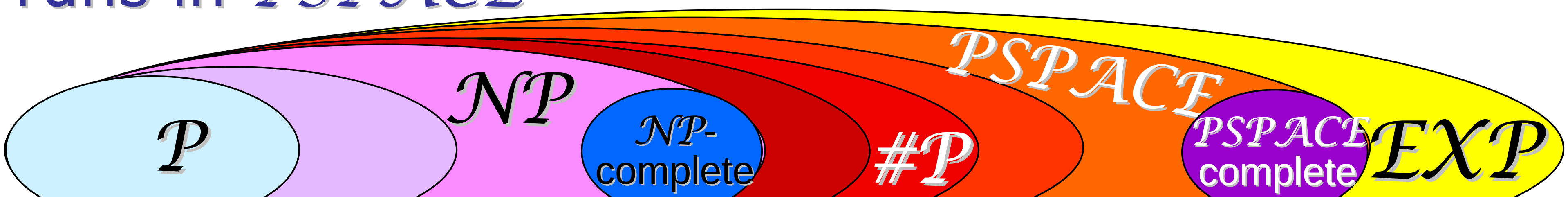
- solution $z()$ exists according to *Peano's Theorem*
- but is **not** unique; [Pour-El&Richards'79]

  In general **none** of the solutions is computable!

Suppose f is Lipschitz-continuous: $\mu(n) = n+O(1)$

- solution is unique according to *Picard-Lindelöf*!
- common approach *Euler's Method/Runge Kutta*:
- $\delta \approx 2^{-\mathrm{poly}(n)}$ ⇒ runs in $\mathcal{PSPACE}$

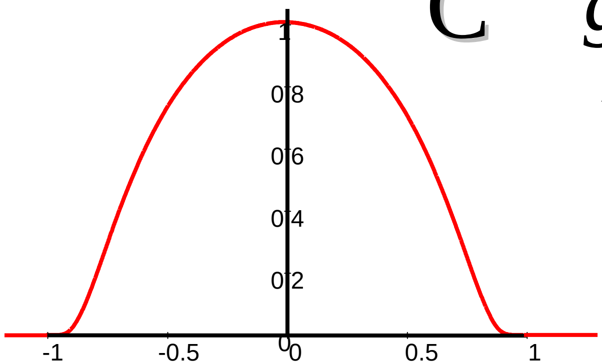
$$z(t+\delta) \approx z(t) + \delta\cdot\dot{z}(t) = z(t) + \delta\cdot f(z,t)$$



'Max is $\mathcal{NP}$-hard'

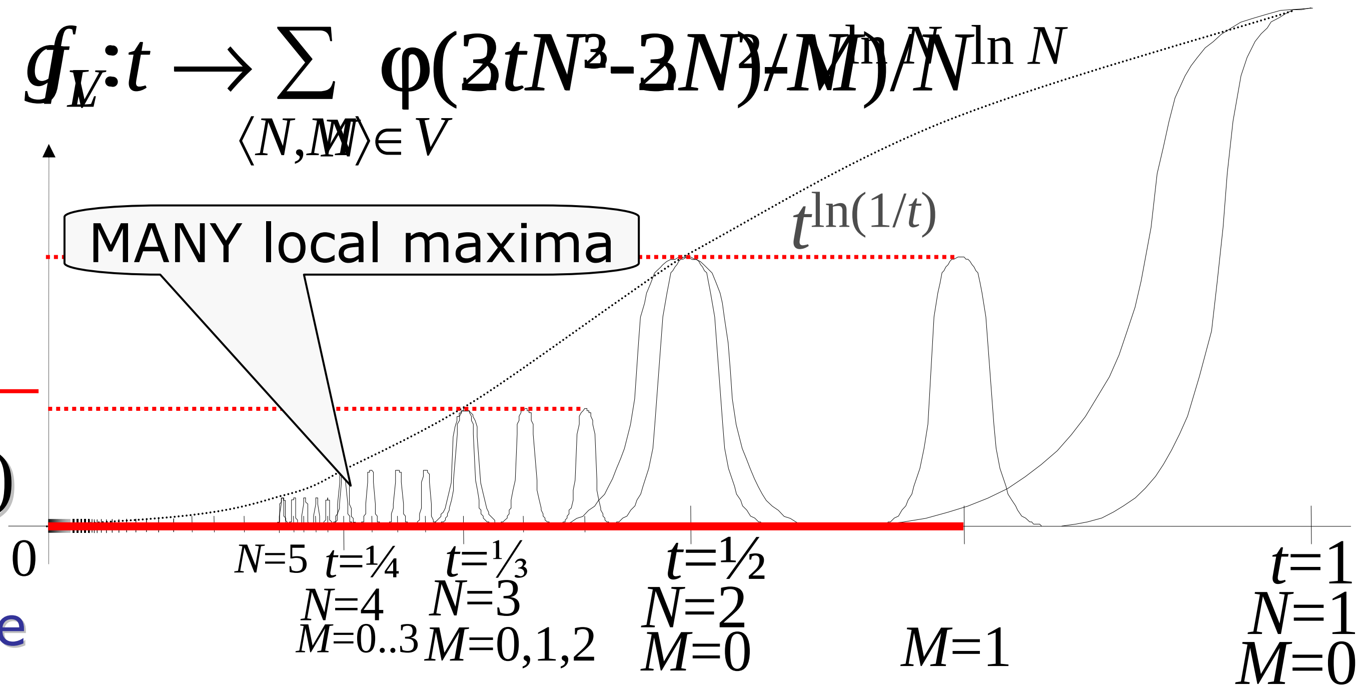
$\mathcal{NP} \ni L = \{\, N\in\mathbb{N} \mid \exists M<N: \langle N,M\rangle \in V \,\},\ V\in\mathcal{P}$ polytime

$C^\infty \quad g_V: t \to \sum_{\langle N,M\rangle\in V} \varphi(3tN^3-3N^3-M)/N^{\ln N}$



$\varphi(t) = \exp(-t^2/1-t^2)$

C^∞ 'pulse' function
polytime computable

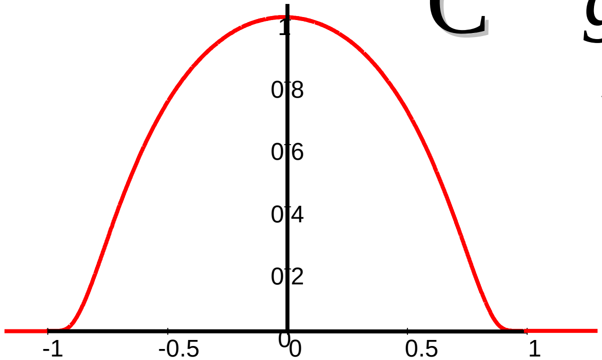


To every $L\in\mathcal{NP}$ there exists a polytime C^∞ function $g_L:[0;1]\to\mathbb{R}$ such that it holds:
$\mathrm{Max}(g_L): x\to\max\{\, g_L(t): t\le x \,\}$ polytime $\Rightarrow L\in\mathcal{P}$

Integration is $\#\mathcal{P}/\#\mathcal{P}_1$-hard

$$\#\mathcal{P} \ni \psi : \mathbb{N} \ni N \to \#\{\, M<N: \langle N,M\rangle \in V \,\},\ V \in \mathcal{P}$$

C^∞ $\quad g_V : t \to \sum_{\langle N,M\rangle\in V} \varphi(3tN^3-3N^2-M)/N^{\ln N}$



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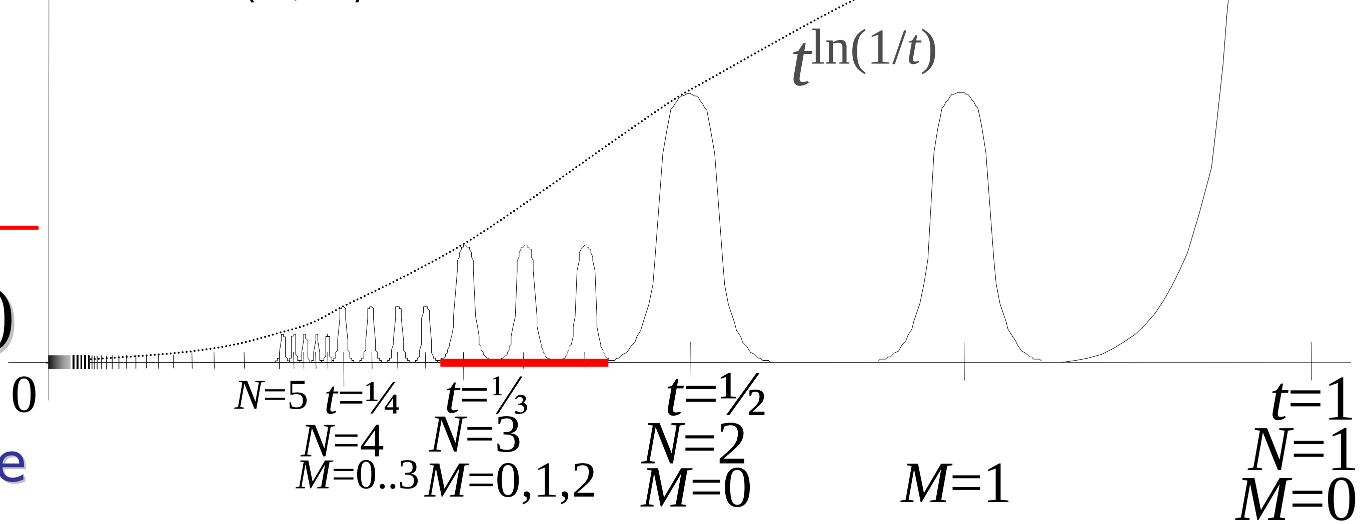


To every $\psi\in\#\mathcal{P}$ there exists a polytime C^∞ function $g_V:[0;1]\to\mathbb{R}$ such that it holds:
$\int g_V:[0;1]^2\ni(x,y)\to\int_x^y g_V(t)\,dt$ polytime $\Rightarrow \psi\in\mathcal{FP}$

Complexity Conjectures in Numerics

KAIST
CS700 M. Ziegler

- $\mathrm{Max}(f): x \to \max\{\, f(t): t \le x \,\}$ characterizes $\mathcal{NP}$
- $\int f: x \to \int_0^x f(t)\,dt$ characterizes $\#\mathcal{P}$
- $\int^1 f: x \to \int_0^1 f(t)\,dt$ characterizes $\#\mathcal{P}_1$
- odesolve: $C^1([0;1]\times[-1;1])\ni f \to z()$: $\dot z(t)=f(t,z),\ \ z(0)=0.$ [Kawamura'10] characterizes $\mathcal{PSPACE}$
- Solution to Poisson's Equation is classical & characterizes $\#\mathcal{P}$
  $\Delta u = f$ on $B_2(\mathbf{0},1)$
  $u = 0$ on $\partial B_2(\mathbf{0},1)$
  [Kawamura,Steinberg,Z.2017]
- Solution to Heat Equation characterizes $\#\mathcal{P}_1$
  $u_t=\Delta u$, $u(0) = f$ on $[0,1)^d$ periodic
  [Koswara,Pogudin,Selivanova,Z. 2020]

Poisson's Equation

Partial differential equation:

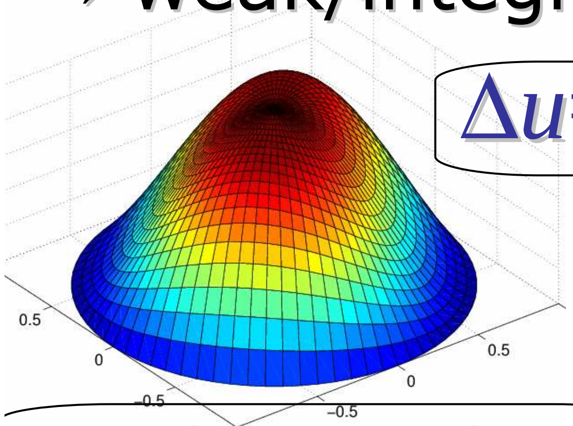
involved solvability [Lewy'57ff],

Navier-Stokes' *Millennium Problem*!

[Hadamard]
Well-posed:
- existence in a function class
- uniqueness in said class
- continuous dependence

[Petrini'1908] Continuous f with no *classical* (i.e. C^2) solution u.

→ weak/integrable/Sobolev solutions

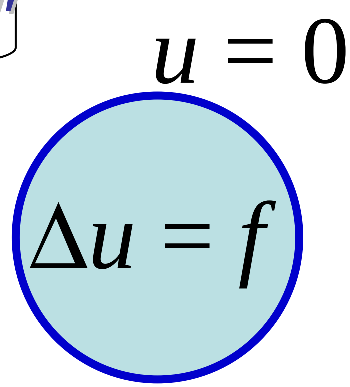


$\Delta u = u_{xx} + u_{yy} + \ldots$

"curvature"

$$\Delta u = f \ \text{ on } \mathrm{B}_d(\mathbf{0},1)$$
$$u = 0 \ \text{ on } \partial\mathrm{B}_d(\mathbf{0},1)$$

Dirichlet Problem



Complexity of Poisson's Equation

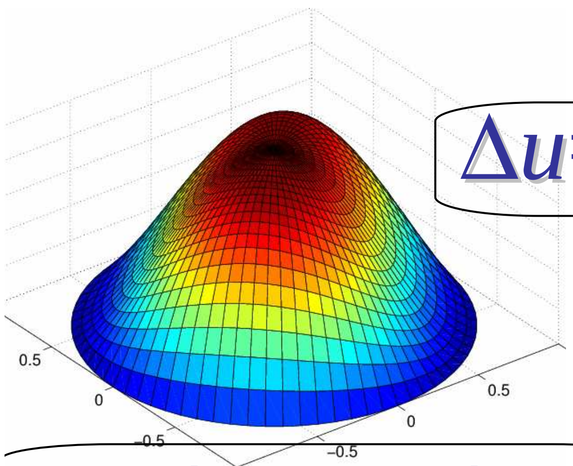
Theorem [Kawamura,Steinberg,Z.2017]

Fix polytime f:$\mathrm{B}_d(\mathbf{0},1)\to\mathbb{R}$, $d>1$.

a) The solution u exists, is classical, and unique.

b) Solution u is polytime if $\mathcal{FP}=\#\mathcal{P}$.

c) There exists a polytime (smooth) f such that u polytime implies $\mathcal{FP}=\#\mathcal{P}$.

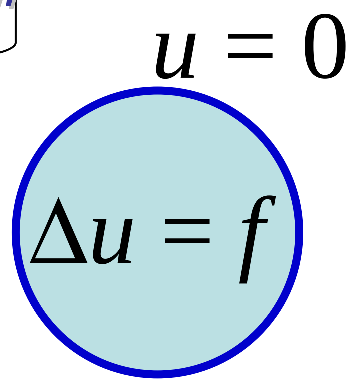


$\Delta u = u_{xx} + u_{yy} + \ldots$

"curvature"

$$\Delta u = f \text{ on } \mathrm{B}_d(\mathbf{0},1)$$
$$u = 0 \text{ on } \partial\mathrm{B}_d(\mathbf{0},1)$$

Dirichlet Problem

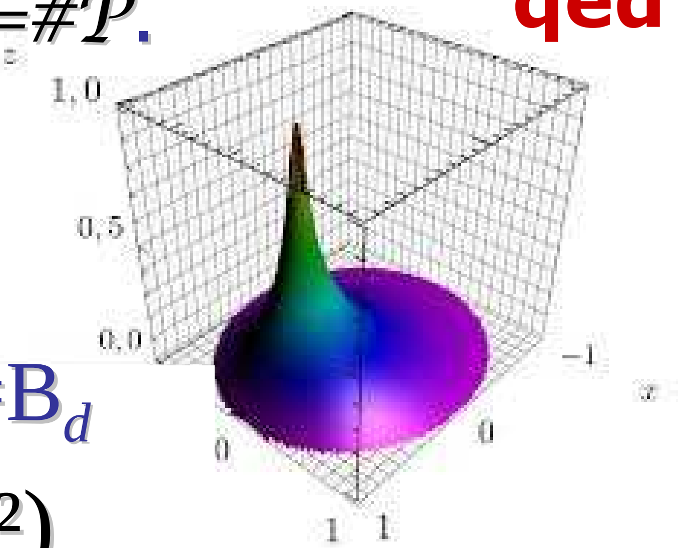


Proof (Sketch) of Claim (b)

Recall: *Indefinite Riemann* integration is "in $\#\mathcal{P}$".

*Im*proper integral: truncate and estimate error.

b) Solution u is polytime if $\mathcal{FP}=\#\mathcal{P}$. **qed**



Fact: $u(\underline{x}) = \int_B G_B(\underline{x},\underline{y})\cdot f(\underline{y})\,d\underline{y}$

with the *Green's Function* for $B=B_d$

$$G_B(\underline{x},\underline{y}) = \Gamma(|\underline{x}-\underline{y}|) - \Gamma(|\underline{x}|\cdot|\underline{y}-\underline{x}/|\underline{x}|^2)$$

and *Fundamental Solution* Γ of $\Delta u = 0$:

$\Gamma(r) = \ln(1/r) / 2\pi$ in dim d=2

$\Gamma(r) = C(d)/r^{d-2}$ in dim d>2

$\Delta u = f$

Proof (Sketch) of Claim (c)

Theorem [Kawamura,Steinberg,Z.2017]

c) There exists a polytime (smooth) $f{:}\mathrm{B}_d(\mathbf{0},1)\to\mathbb{R}$
such that u polytime implies $\mathcal{FP}=\#\mathcal{P}$.

Recall: There exists a polytime C^∞ $g{:}[0;1]\to\mathbb{R}$ such that $\int_0^y g(t)\,\mathrm{d}t$ polytime $\Rightarrow$ $\mathcal{FP}=\#\mathcal{P}$.

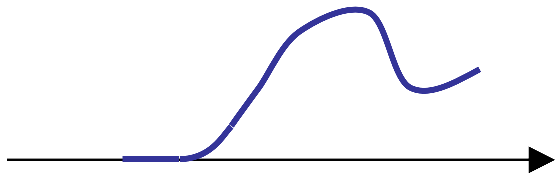
wlog $g|_{[0;¼]}\equiv 0$

For *radially* symmetric $f(\underline{x})=f(|\underline{x}|)$,

$$f(r) = \Delta u(r) = (r^{d-1}\cdot u'(r))' \,/\, r^{d-1}.$$

$f(r) := g(r)/r^{d-1}$ <u>polytime</u>

$\Rightarrow\ r^{d-1}\cdot u'(r) = \int g(r)\,\mathrm{d}r$



$\Delta u = f$

Heat Equation

Fourier *Ansatz*

$$u_0(x) = \sum_{\underline{k}\in\mathbb{Z}^d} c_{\underline{k}}\cdot e^{2\pi i\langle \underline{k}\cdot x\rangle}$$

$$u(\underline{x},t) = \sum_{\underline{k}\in\mathbb{Z}^d} c_{\underline{k}}\cdot e^{2\pi i\langle \underline{k}\cdot x\rangle - 4\pi t k^2}$$

$$c_{\underline{k}} = \int u_0(\underline{x})\cdot e^{-2\pi i\langle \underline{k}\cdot x\rangle}$$

fundamental solution

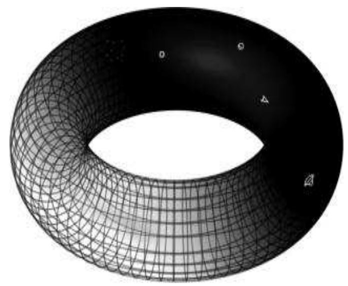
Fact: For continuous u_0, $(c_{\underline{k}})$ is bounded and $u(\underline{x},t)$ is analytic in $\underline{x}$ for every $t>0$.

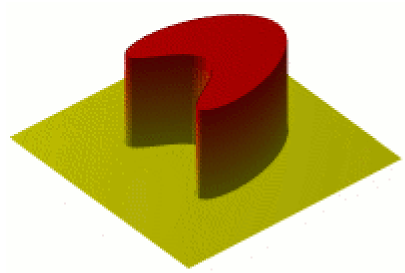
Theorem: Suppose u_0 is polytime.

a) $(c_{\underline{k}})$ is computable "in $\#\mathcal{P}_1$"

b) So is $u=u(\underline{x},t)$ for every fixed $t>0$.

c) $\#\mathcal{P}_1$ is optimal for $u=u(\underline{x},t)$

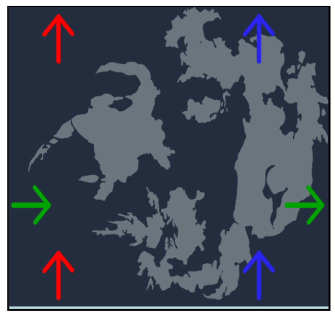




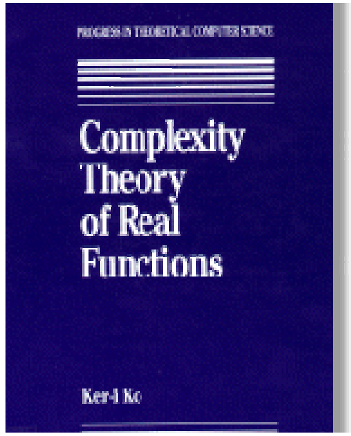
$$u_t = \Delta u \quad \text{on } [0,1)^d \bmod 1$$

$$u(0) = u_0$$

periodic



V. Complexity Theory over the Reals



- *Polytime*-computable real numbers
- *Polytime*-computable real functions
- *Quantitative* computability ⇔ quantitative continuity
- Operations that preserve *polytime* functions
- (Strongly) *polytime*-computable real sequences
- Polytime analytic functions ⇔ polytime Taylor series
- Operations that preserve *polytime* <u>analytic</u> functions

V. Complexity Theory over Reals (2)

- *Parametric* Maximization "*in* $\mathcal{NP}$"
- *Indefinite* Riemann Integration "*in* **#**$\mathcal{P}$"
- *Definite* Riemann Integration "*in* **#**$\mathcal{P}_1$"
- ODESOLVE "*in* $\mathcal{PSPACE}$"
- Parametric Maximization is $\mathcal{NP}$-"*complete*"
- In/definite integration is $\#\mathcal{P}/\#\mathcal{P}_1$-"*complete*"
- Complexity of PDEs: Poisson and Heat Equation
- More numerical characterizations of discrete complexity classes