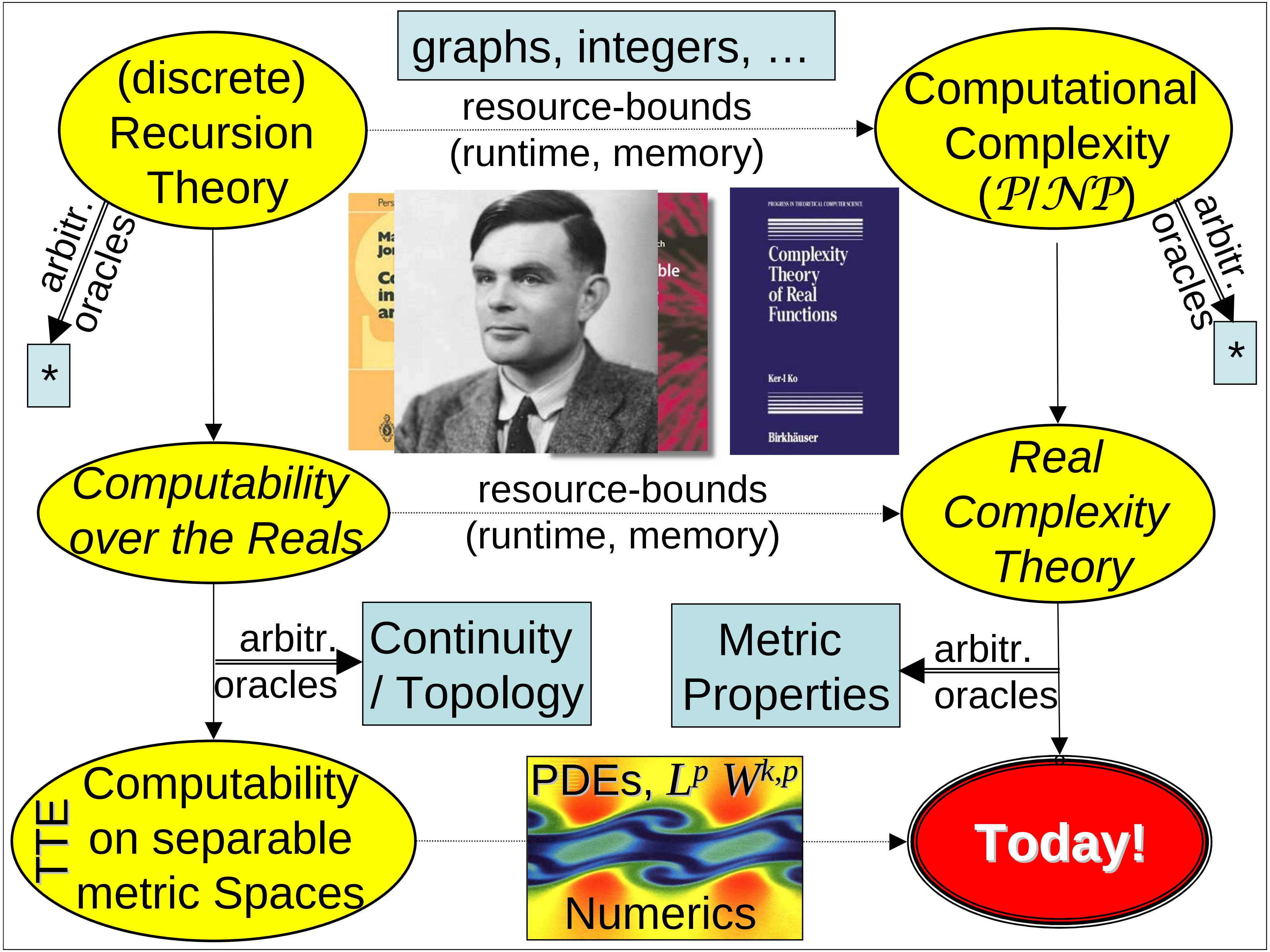
VI. Complexity on Metric Spaces

- Complexity on (elements of) metric spaces
- Complexity of functions (between metric spaces)
- Complexity and Quantitative Continuity?
- *Polynomial* Admissibility
- Polynomial *Main Theorem*
- Entropy = quantitat. compactness
- *Standard* representation
- 2nd-order Basic Space

qua**l**itative	qua**nt**itative
computability	complexity
topology	metric
(uniformly) continuous	modulus of continuity
compact	entropy
continuous image of compact	*Steinberg's Lemma*
equilogical space	compact *ultrametric*



graphs, integers, …
(discrete) Recursion Theory
resource-bounds (runtime, memory)
Computational Complexity ($\mathcal{P}/\mathcal{NP}$)
arbitr. oracles
*
arbitr. oracles
*
PROGRESS IN THEORETICAL COMPUTER SCIENCE
Complexity Theory of Real Functions
Ker-I Ko
Birkhäuser
Computability over the Reals
resource-bounds (runtime, memory)
Real Complexity Theory
arbitr. oracles
Continuity / Topology
Metric Properties
arbitr. oracles
TTE
Computability on separable metric Spaces
PDEs, L^p $W^{k,p}$
Numerics
Today!

Complexity on Metric Spaces

Recall: For $t:\mathbb{N}\to\mathbb{N}$, computing $\underline{u}\in\mathcal{B}$ in time t means:

output $u_0,\ldots,u_{n-1}$ in $\leq t(n)$ steps.

Computing $r\in\mathbb{R}$ in time t:

fin.prefix of *name*

output $a_0,\ldots a_{n+1}$ in $t(n)$ steps st. $|r-a_m/2^m|\leq 2^{-m}$

Def: Fix representation $\xi:\to(X,d)$. Compute $x\in X$ in time t:

Output $u_0,\ldots u_\mu$ in $\leq t(\boldsymbol{n})$ steps s.t. $=:\underline{u}$

(i) there exist $u_{\mu+1},\ldots$ s.t. $\xi(u_0,\ldots u_\mu,u_{\mu+1},\ldots)=x$

(ii) $d(x,\xi(\underline{u}))\leq 2^{-n}$ for all $u_{\mu+1},\ldots$ with $\underline{u}\in\mathrm{dom}(\xi)$

$\mu=\mu(n)$ is a modulus of continuity of ξ

Modulus $\mu:\mathbb{N}\to\mathbb{N}$ of $h:(A,d)\to(B,e)$:
$d(a,a')\leq 2^{-\mu(n)} \Rightarrow e(h(a),h(a'))\leq 2^{-n}$

Baire space $\mathcal{B}=\mathbb{Z}^{\mathbb{N}}$,
metric $D(\underline{u},\underline{v})=2^{-\min\{n:u_n\neq v_n\}}$

Dyadic representat. $\delta:\subseteq\mathcal{B}\ni(a_n)\to r\in\mathbb{R}$ s.t. $|r-a_n/2^n|\leq 2^{-n}$

Complexity of Functions

Recall: For $t:\mathbb{N}\to\mathbb{N}$, computing $F:\subseteq\mathcal{B}\to\mathcal{B}$ in time t means:

On input $\underline{u}\in\mathrm{dom}(F)$, compute $\underline{v}=F(\underline{u})$ in time t.

Compute $f:\subseteq\mathbb{R}\to\mathbb{R}$ in time t: regardless of $x\in\mathrm{dom}(f)$ and $\underline{a}$

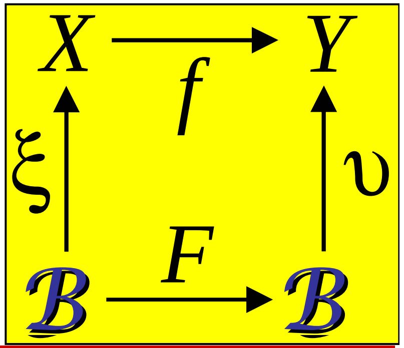
input $(a_m)\subseteq\mathbb{Z}$ s.t. $|x-a_m/2^m|\leq 2^{-m}$, compute $y=f(x)$ in time t.

Def: Fix representation $\upsilon:\to(Y,e)$. Compute $y\in Y$ in time t:

Output $v_0,\ldots v_\mu$ in $\leq t(n)$ steps s.t. $=:\underline{v}$

(i) there exist $v_{\mu+1},\ldots$ s.t. $\xi(v_0,\ldots v_\mu,v_{\mu+1},\ldots)=y$

(ii) $e(y,\upsilon(\underline{v}))\leq 2^{-n}$ for all $v_{\mu+1},\ldots$ with $\underline{v}\in\mathrm{dom}(\upsilon)$



Def: Fix representations $\xi:\to(X,d)$, $\upsilon:\to(Y,e)$. (ξ,υ)-compute $f:X\to Y$ in time t: on input of any ξ–name of $x\in\mathrm{dom}(f)$, compute $y=f(x)$ in time t.

regardless of $x\in\mathrm{dom}(f)$

A (ξ,υ)-realizer of $f:X\to Y$ is a $F:\mathrm{dom}(\xi)\to\mathrm{dom}(\upsilon)$ s.t. $f\circ\xi\sqsubseteq\upsilon\circ F$.

Complexity and Continuity?

Recall: $F:\subseteq C\to C$ computable in time t

$\Rightarrow t$ is a modulus of continuity of F

Theorem: $f:\subseteq\mathbb{R}\to\mathbb{R}$ has <u>modulus</u> $\mathrm{poly}(\mu(\mathrm{poly}(n)))$

$\Leftrightarrow$ has a (δ,δ)-realizer <u>with modulus</u> $\mathrm{poly}(\mu(\mathrm{poly}(n)))$.

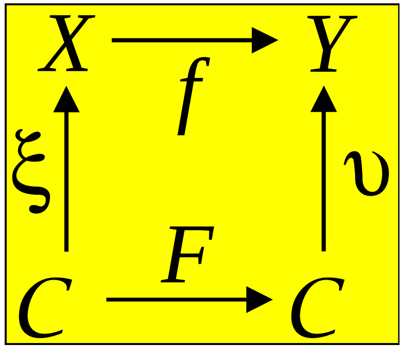
Recall: Representation ξ of X is admissible (standard) if

(i) is continuous and

(ii) <u>every</u> continuous ~~surjective~~ $\xi':\to X$

satisfies $\xi'\preceq\xi$.

(Very) different from computing a (ξ,υ)-realizer of f in time t!



Def: Fix representations $\xi:\to(X,d)$, $\upsilon:\to(Y,e)$.

(ξ,υ)-compute $f:X\to Y$ in time t: on input of any ξ–name of $x\in\mathrm{dom}(f)$, compute $y=f(x)$ in time t.

regardless of $x\in\mathrm{dom}(f)$

$\xi'\preceq\xi$ means $\xi'\sqsubseteq\xi\circ F$ for some continuous $F:\mathrm{dom}(\xi')\to\mathrm{dom}(\xi$

Quantitative Admissibility

Lemma: i) $\delta :\subseteq C \to [0;1]$ has <u>modulus</u> $\mathrm{poly}(n)$

ii) every $\xi' :\to [0;1]$ has: $\xi' \preceq_p \delta$

***Real* Main Theorem:** $f :\subseteq \mathbb{R} \to \mathbb{R}$ has <u>modulus</u> $\mathrm{poly}(\mu(\mathrm{poly}(n)))$

$\Leftrightarrow$ has a (δ,δ)-realizer <u>with modulus</u> $\mathrm{poly}(\mu(\mathrm{poly}(n)))$.

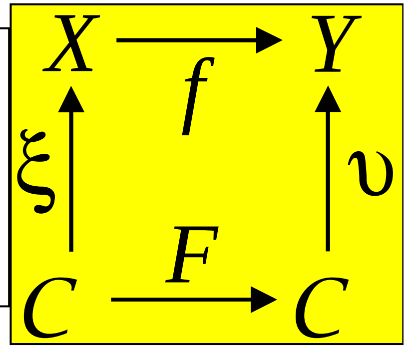
Recall: Representation ξ of X is admissible (standard) if

(i) is continuous and

(ii) every continuous ~~surjective~~ $\xi' :\to X$ satisfies $\xi' \preceq \xi$.

transitive

Recall: If $f : X \to Y$ has μ and $g : Y \to Z$ has ν, then $g \circ f$ has $\mu \circ \nu$.



Recall *Main Theorem*: Fix admissible $\xi :\to X$ and $\upsilon :\to Y$.

$f : X \to Y$ is continuous **iff** it has a continuous (ξ,υ)-realizer.

$\xi' \preceq_p \xi$ means $\xi' \sqsubseteq \xi \circ F$ for $F : \mathrm{dom}(\xi') \to \mathrm{dom}(\xi)$ <u>with modulus</u> ν

st. $\nu(\mu(n)) \leq \mu'(\mathrm{poly}(n))$ for the <u>minimal</u> moduli μ, μ' of ξ, ξ'

Polyn. Admissibility / *Main Theorem*

Lemma: i) $\delta:\subseteq C\to[0;1]$ has modulus $\mathrm{poly}(n)$

ii) every $\xi':\to[0;1]$ has: $\xi'\preceq_p\delta$

δ is polyn. *standard*

Lemma [Steinberg'17]: If $f:A\to B$ has modulus μ and A has entropy η_A, then $\mathrm{range}(f)\subseteq B$ has entropy $\leq\eta_A\circ\mu$.

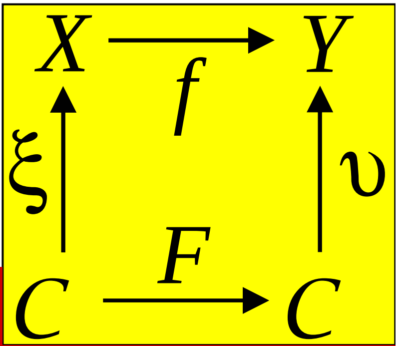
$f\circ\xi \sqsubseteq$ $\sqsubseteq \upsilon\circ F$

Def: Call $\xi:\subseteq C\to X$ polynomially admissible (standard) if

(i) has modulus $\mu_\xi(n)\leq\mathrm{poly}(\eta(\mathrm{poly}(n)))$

(ii) every ~~surjective~~ $\xi':\to X$ satisfies $\xi'\preceq_p\xi$.

$\eta=\eta_X:\mathbb{N}\to\mathbb{N}$ denotes the entropy of (X,d).



Polyn. *Main Theorem*: a) Every compact (Y,e) with $\eta_Y(n)\geq\Omega(n^\varepsilon)$ has a polynom. *standard* representation υ.

b) If $f:X\to Y$ has modulus μ_f, then it has a (ξ,υ)-realizer F with modulus μ_F s.t. $\mu_F\circ\eta_Y\leq\mu_f\circ\mathrm{poly}\circ\eta_X\circ\mathrm{poly}$.

$\upsilon' := \xi\circ f$

c) If $f:X\to Y$ has a (ξ,υ)-realizer F with modulus μ_F, then f has modulus μ_f s.t. $\mu_f\circ\eta_X\leq\mu_F\circ\mathrm{poly}\circ\eta_Y\circ\mathrm{poly}$.

$\eta\leq\mu_\xi$

Entropy (=Quantitative Compactness)

Lemma: i) $\delta:\subseteq C\to[0;1]$ has <u>modulus</u> $\mathrm{poly}(n)$

Def [Kolmogorov'59]: Entropy of (X,d) is $\eta:\mathbb{N}\to\mathbb{N}$ s.t. X can be covered by $2^{\eta(n)}$, but not $2^{\eta(n)-1}$, balls of radius 2^{-n}.

Lemma [Steinberg'17]: If $f:A\to B$ has modulus μ and A has entropy η_A, then $\mathrm{range}(f)\subseteq B$ has entropy $\leq\eta_A\circ\mu$.

(i) Represent. $\xi:\subseteq C\to X$ has modulus $\mu_\xi(n)\leq\mathrm{poly}(\eta(\mathrm{poly}(n)))$

$\eta=\eta_X:\mathbb{N}\to\mathbb{N}$ denotes the entropy of (X,d).

Necessary
$\eta\leq\mu_\xi$

<u>**Polyn.**</u> ***Main Theorem*: a)** Every compact (X,d) with $\eta_X(n)\geq\Omega(n^\varepsilon)$ <u>has</u> a polynom.*standard* representation ξ.

Examples: a) $[-2^k;2^k]^d$ has entropy $\eta(n)=\Theta(d\cdot(n+k))$

b) $C=\{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$ has entropy $\eta(n)=n$

c) $X' = \mathrm{Lip}_1(X,C)$ has entropy $\eta'(n)=2^{\mathrm{poly}(n+\eta(\Theta(n)))}$.

Constructing *Standard* Representation

Def [Kolmogorov'59]: Entropy of (X,d) is $\eta:\mathbb{N}\to\mathbb{N}$ s.t. X can be covered by $2^{\eta(n)}$, but not $2^{\eta(n)-1}$, balls of radius 2^{-n}.

For n, number centers c_N of covering balls: $\mathbb{N}\ni N<2^{\eta(n)}$.

Name of $x\in X$: list N_{n+1} of indices of balls $B(c_N,2^{-n-1})\ni x$

N_n in binary: length $\eta(n+1)$ $\Rightarrow$ modulus $\sum_{m\leq n+1}\eta(n+1)$

(i) Represent. $\xi:\subseteq C\to X$ has modulus $\mu_\xi(n)\leq\text{poly}(\eta(\text{poly}(n)))$

Necessary
$\eta\leq\mu_\xi$

Polyn. ***Main Theorem*****: a)** Every compact (X,d) with $\eta_X(n)\geq\Omega(n^\varepsilon)$ has a polynom.*standard* representation ξ.

$$\sum_{m\leq n+1}\eta(n+1)\leq(n+1)\cdot\eta(n+2)\leq(\eta(n+2))^{1+1/\varepsilon}$$

b) $C=\{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$ has entropy $\eta(n)=n$

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2*nd-Order* Basic Space

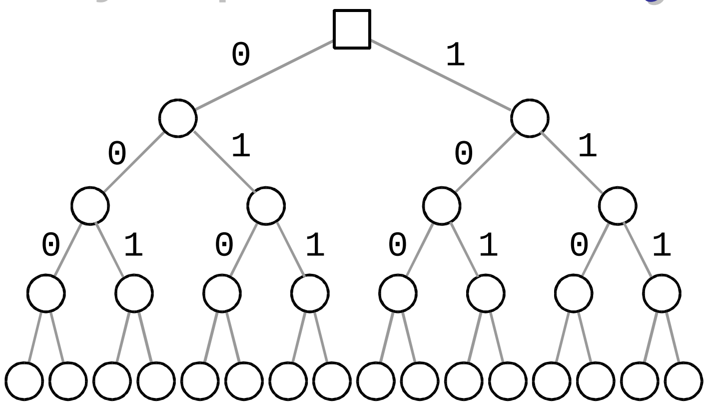
Replace basic space $C=\{\mathtt{0},\mathtt{1}\}^{\mathbb{N}} = \{\mathtt{0},\mathtt{1}\}^{\{\mathtt{1}\}^*}$ (seq.access)

access u_m in $O(\log m)$ steps | access u_m in m steps **SHIFT**

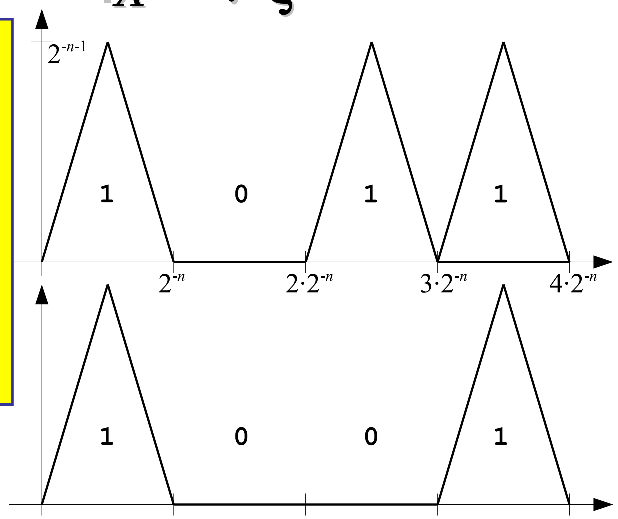
with $C' = \mathrm{Lip}_1(C,C) = \{\mathtt{0},\mathtt{1}\}^{\{\mathtt{0},\mathtt{1}\}^*}$ rand./indirect access

Lemma [Steinberg'17]: If $f{:}A\to B$ has modulus μ and A has entropy η_A, then $\mathrm{range}(f)\subseteq B$ has entropy $\leq\eta_A\circ\mu$.

Any representation $\xi{:}\subseteq C\to X$ has modulus $\eta_X \leq \mu_\xi$



"Application"
$C'\times C \ni (x',x)$
$\to x'(x) \in C$
now in polytime



b) $C=\{\mathtt{0},\mathtt{1}\}^{\mathbb{N}}$ has entropy $\eta(n)=n$

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