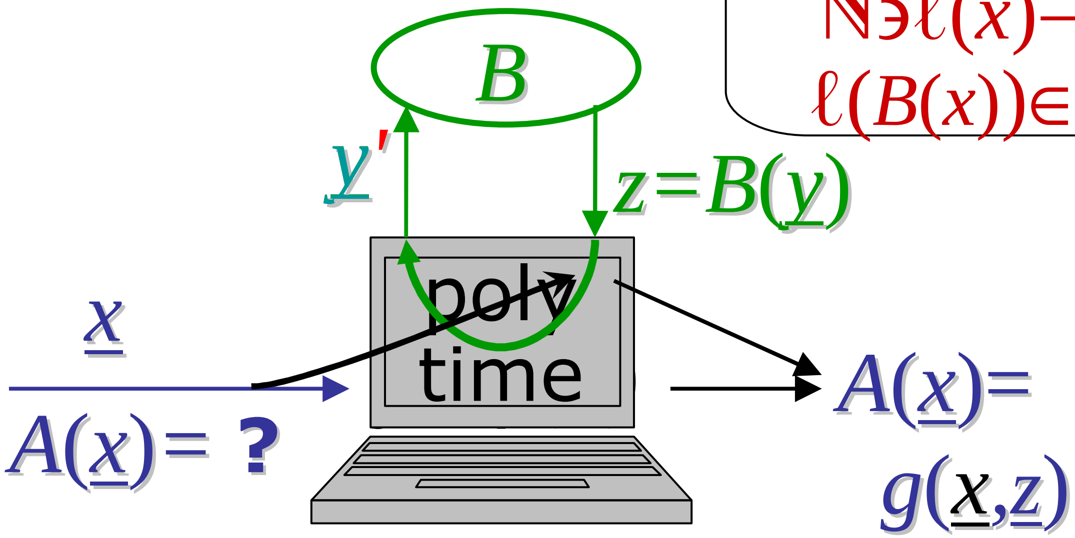
- Computing Integer Function*al*s/Operators
- Various Polynomial Resource Bounds
- Degree of Polynomials
- Properties of 2$^{nd}$-order Degree

# Polytime Turing reduction $A \leqslant_p^{T[1]} B$

Let $B:\mathbb{N}\to\mathbb{N}$ vary ⇒ function**al** $A: (B,x) \longrightarrow A(B)(x)$

Runtime depends on $x\in\mathbb{N}$, — length $\ell(x)\in\mathbb{N}$

and on $B:\mathbb{N}\to\mathbb{N}$! — length $\Lambda(B)$: $\mathbb{N}\ni\ell(x)\longrightarrow\ell(B(x))\in\mathbb{N}$




- receive input $x$
- process $x\to y=f(x)$
- *query* "$B(y)=z$"
- process *reply* $z$

  (and input $x$)
- answer "$A(x)$"

For $A,B:\mathbb{N}\to\mathbb{N}$ write $A\leqslant_p^{T[1]} B$ if $A = g \circ B \circ f$
for polytime $f:\mathbb{N}\to\mathbb{N}$, $g:\mathbb{N}^2\to\mathbb{N}$

# Polynomial Resource Bounds



- closed under + and × $\Rightarrow$ also under composition ∘

*Uni*variate polynomial ring $\mathbb{N}[N]$ :

= least set of *terms* $p$ (=formal expressions)

- containing constants 0,1 and variable $N$

Interpret $N\in\mathbb{N}$ $\Rightarrow$ $\mathbb{N}[N] \ni p \to p \in \mathbb{N}^{\mathbb{N}}$

$\mathcal{P}$, $\mathcal{FP}$, $\mathcal{PSPACE}$

*Bi*variate polynomials $q\in\mathbb{N}[N,K]$ :

Least set of *terms* contain. 0,1 and variable**s** $N,K$

Interpret $N,K\in\mathbb{N}$ $\Rightarrow \mathbb{N}[N,K] \ni q \to q \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}}$

$\mathcal{FPT}$

2nd-order polynomials $P\in\mathbb{N}[N,\Lambda]$ :

Least set of *terms* containing 0,1 and variable**s** $N,\Lambda$

Interpret $N\in\mathbb{N}$, $\Lambda\in\mathbb{N}^{\mathbb{N}}$ $\Rightarrow \mathbb{N}[N,\Lambda] \ni P \to P \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}^{\mathbb{N}}}$

- terms induce functions: bounds in complexity theory

# Degree of 2nd-order Polynomials



- closed under + and × $\Rightarrow$ $\deg(p\circ q) = \deg(p) \times \deg(q)$

**Bi**variate polynomials $q\in\mathbb{N}[N,K]$ : total-deg($K$)=1

$\deg:\mathbb{N}[N]\to\mathbb{N}\cup\{-\infty\}$, $\deg(0)=-\infty$, $\deg(1)=0$, $\deg(N)=1$,

$\deg(p+q)=\max(\deg(p),\deg(q))$, $\deg(p\times q)=\deg(p)+\deg(q)$

$\mathrm{DEG}:\mathbb{N}[N,\Lambda]\to(\mathbb{N}\cup\{-\infty\})^{\mathbb{N}}$

$\mathrm{DEG}(\Lambda(P))(M) = M\cdot\mathrm{DEG}(P)$

$\mathrm{DEG}(0)=-\infty$, $\mathrm{DEG}(1)=0$, $\mathrm{DEG}(N)=1$,

$\mathrm{DEG}(P+Q)=\max(\mathrm{DEG}(P),\mathrm{DEG}(Q))$,

$\mathrm{DEG}(P\times Q)=\mathrm{DEG}(P)+\mathrm{DEG}(Q)$

$\mathrm{DEG}()=\max(\ 3+M\ ,\ 1+M\cdot\max(3+M\cdot 2,1+3\cdot M)\ )$

2nd-order polynomials $P\in\mathbb{N}[N,\Lambda]$

Least set of *terms* containing 0,1 … iable**s** $N,\Lambda$

**Interpret** $N\in\mathbb{N}$, $\Lambda\in\mathbb{N}^{\mathbb{N}}$ $\Rightarrow$ $\mathbb{N}[N,\Lambda]$ … $\mathbb{N}\times\mathbb{N}^{\mathbb{N}}$

**Example:** $N\times\ \Lambda(\ N^3\times\Lambda(N+N^2) + N\times\Lambda^3(N)\ ) + N^3\times\Lambda(N)$

# Properties of $2^{nd}$-order Degree



- closed under + and × ⇒ $\deg(p \circ q) = \deg(p) \times \deg(q)$

**Theorem:** Fix $2^{nd}$-order polynomial $P \in \mathbb{N}[N,\Lambda]$.

**a)** For every $d \in \mathbb{N}$, $P(N, N \to N^d) \in \mathbb{N}[N]$: $1^{st}$ ord.polynom

**b)** For all $p \in \mathbb{N}[N]$, $\mathrm{DEG}(P)(\deg p) = \deg P(N,p)$.

**c)** For all sufficiently large $d \in \mathbb{N}$, $\mathrm{DEG}(P)(d) \in \mathbb{N}[d]$

**d)** $\mathrm{DEG}(P(Q,\bullet)) = \mathrm{DEG}(P) \times \mathrm{DEG}(Q)$,

**e)** $\mathrm{DEG}(P(\bullet,Q)) = \mathrm{DEG}(P) \circ \mathrm{DEG}(Q)$

$P(\bullet,Q) := (N,\Lambda) \to P(N, M \to Q(M,\Lambda))$

$P(Q,\bullet) := (N,\Lambda) \to P(Q(N,\Lambda),\Lambda)$

Interpret $N \in \mathbb{N}$, $\Lambda \in \mathbb{N}^{\mathbb{N}}$

$\mathrm{DEG}() = \max(\, 3+M\,,\; 1+M \cdot \max(3+M \cdot 2, 1+3 \cdot M)\,)$

$= 1+M \cdot (1+3M)$, $M \geq 2$

| $d$ | 0 | 1 | 2 | 3 |
| --- | --- | --- | --- | --- |
| Deg | 3 | 6 | 15 | 28 |

**Example:** $N \times \Lambda(\, N^3 \times \Lambda(N+N^2) + N \times \Lambda^3(N)\,) + N^3 \times \Lambda(N)$