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Syllabus

3. Sorting

- Specification
- Bubble Sort
- Selection Sort
- Insertion Sort
- Merge Sort
- Quicksort
- Linear-Time Median
- Sorting in Linear Time
- Sorting in Parallel

- Specification
- Primitives: semantics and cost
- Design
- Analysis
- Optimality

of Comparison-Based Sorting

3. Sorting

specification

Specification: Fix set X with total order $\leq$.

Input: $N \in \mathbb{N}$ and finite sequence $x_1, \ldots x_N \in X$ in array $x[1 \ldots N]$

and array $\pi[1 \ldots N] := $ identity permutation of $\{1, \ldots N\}$

Output: Permutation π of $\{1, \ldots N\}$ such that $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Primitives:

ordered comparison "$x[n] \leq x[m]$?"	cost 1
Index integer arithmetic	cost 1
swapping $x[n] \leftrightarrow x[m]$	cost 1
swapping $\pi[n] \leftrightarrow \pi[m]$	cost 1

Preprocessing data in order to accelerate queries.

3. Sorting

Bubble Sort

Input: $N \in \mathbb{N}$ and array $x[1 \ldots N]$

Output: *Permuted* array $x \circ \pi$ s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Primitives:

Comparison "$x[n] \leq x[m]$?"

Swapping $x[n] \leftrightarrow x[m]$

Index integer arithmetic

```
Procedure BubbleSort ( x[N] )
For  m := N  downto 2  do
    For  k := 1 to m−1 do
        If  x[k] > x[k+1]  then
            Swap (x[k] , x[k+1])
        Endif
    Endfor
Endfor
```

6 5 3 1 8 7 2 4

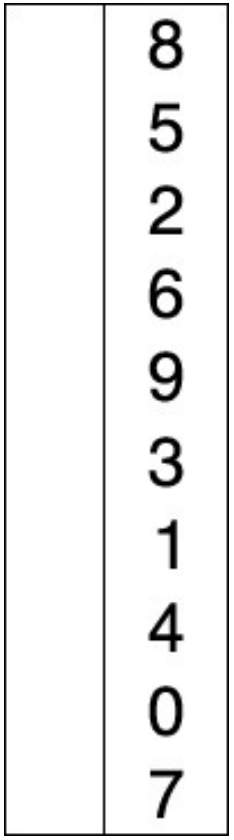
runtime $O(N^2)$

Correctness: $x[1], \ldots x[m] \leq x[m+1] \leq \ldots \leq x[N]$

3. Sorting

Select Sort

```
Procedure SelectSort ( x[N] )
For m := 1 to N−1 do
    min := m;
    For k := min+1 to N do
        If x[k] < x[min] then
            min := k ; Endif
    Swap( x[m], x[min] )
    Endfor
Endfor
```



Input: $N \in \mathbb{N}$ and array $x[1 \ldots N]$

Output: *Permuted* array $x \circ \pi$ s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Primitives:

Comparison "$x[n] \leq x[m]$?"

Swapping $x[n] \leftrightarrow x[m]$

Index integer arithmetic

runtime $O(N^2)$

Correctness: $x[1] \leq \ldots \leq x[m\text{-}1] \leq x[m], \ldots x[N]$

3. Sorting

Insert Sort

6 5 3 1 8 7 2 4

Procedure **InsertSort** ($x[N]$)

For $m := 2$ to N do

$y := x[m]; \quad k := m-1;$

While $k>0$ and $x[k]>y$

$x[k+1] := x[k]$

$k := k-1$

Endwhile

$x[k+1] := y$

Endfor

Input: $N \in \mathbb{N}$ and array $x[1 \ldots N]$

Output: *Permuted* array $x \circ \pi$ s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Primitives:

Comparison "$x[n] \leq x[m]$?"

Swapping $x[n] \leftrightarrow x[m]$

Index integer arithmetic

runtime $O(N^2)$

Correctness: $x[1] \leq \ldots \leq x[m\text{-}1] \leq x[m], \ldots x[N]$

3. Sorting

Merge Sort

Input: $N \in \mathbb{N}$ and array $x[1 \ldots N]$

Output: *Permuted* array $x \circ \pi$ s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Procedure **MergeSort** ($x[N]$)

If $N \leq 1$ return.

$l := \lfloor N/2 \rfloor$; $r := \lceil N/2 \rceil$; array $y[l]$, $z[r]$;

For $m := 1$ to l do $y[m] := x[m]$;

For $m := 1$ to r do $z[m] := x[l+m]$;

MergeSort(y); MergeSort(z);

While $l>0$ and $r>0$ do ; If $z[r] < y[l]$

then $x[l+r] := y[l]$; $l := l-1$

else $x[l+r] := z[r]$; $r := r-1$

While $l>0$ do ; $x[l+r] := y[l]$; $l := l-1$; Endwhile

While $r>0$ do ; $x[l+r] := z[r]$; $r := r-1$; Endwhile

6 5 3 1 8 7 2 4

runtime
$T(N) = 2 \cdot T(N/2) + O(N)$
$\leq O(N \cdot \log N)$

memory
$O(N \cdot \log N)$

3. Sorting

Quick Sort

Procedure **QuickSort** ($x[]$; l,r)

If $l \geq r$ return. // $x[l]\ldots x[r]$ sorted

$s := x[\lfloor (l+r)/2 \rfloor] \in X$ // sample pivot

$a := l$; $b := r$; While $a < b$ do

  While $a<b$ and $x[a] < s$ do $a := a+1$ Endwhile;

  While $a<b$ and $x[b] \geq s$ do $b := b-1$ Endwhile;

  Swap($x[a]$,$x[b]$);

  Endwhile ; // $l \leq a = b \leq r$

QuickSort (x , l , $a-1$);

QuickSort (x , b , r);

Idea: $s{:=}x[p]$ for ***random*** $p \in [l..r]$

Input: $N \in \mathbb{N}$ and array $x[1\ldots N]$

Output: *Permuted* array $x \circ \pi$ s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

$T(N) \leq O(N) + T(N-2) + T(2) = O(N^2)$

l	$a=b$	r

QuickSort is in worst-case as bad as **`BubbleSort`**

→ *randomized* algorithms

3. Sorting

continued: Quick Sort

Input: $N \in \mathbb{N}$ and array $x[1 \ldots N]$

Output: *Permuted* array $x \circ \pi$ s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Procedure **QuickSort** ($x[]$; l,r)

If $l \geq r$ return. // $x[l] \ldots x[r]$ sorted

$y := x[\lfloor (l+r)/2 \rfloor] \in X$ // sample pivot

$a := l$; $b := r$; While $a < b$ do

While $a<b$ and $x[a] < y$ do $a := a+1$ Endwhile;

While $a<b$ and $x[b] \geq y$ do $b := b-1$ Endwhile;

Swap($x[a]$,$x[b]$);

Endwhile ; Return a;

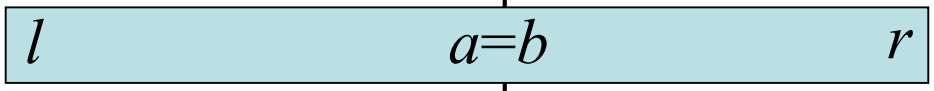
QuickSort (x , l , $a-1$);

QuickSort (x , b , r);

$$T(N) = T(N \cdot \varepsilon) + T(N \cdot (1-\varepsilon)) + O(N)$$

$$\leq O(N \cdot \log N)$$

if $\varepsilon \in (0,1)$ ***fixed***



$$\varepsilon \cdot (r-l+1) \leq a-l \leq (1-\varepsilon) \cdot (r-l+1)$$

$$\varepsilon \cdot (r-l+1) \leq r-a \leq (1-\varepsilon) \cdot (r-l+1)$$

$$c \cdot N \cdot \log(N) =: T(N) = c \cdot N \cdot \varepsilon \cdot \log(N \cdot \varepsilon) + c \cdot N \cdot (1-\varepsilon) \cdot \log(N \cdot (1-\varepsilon)) + N$$

Ansatz

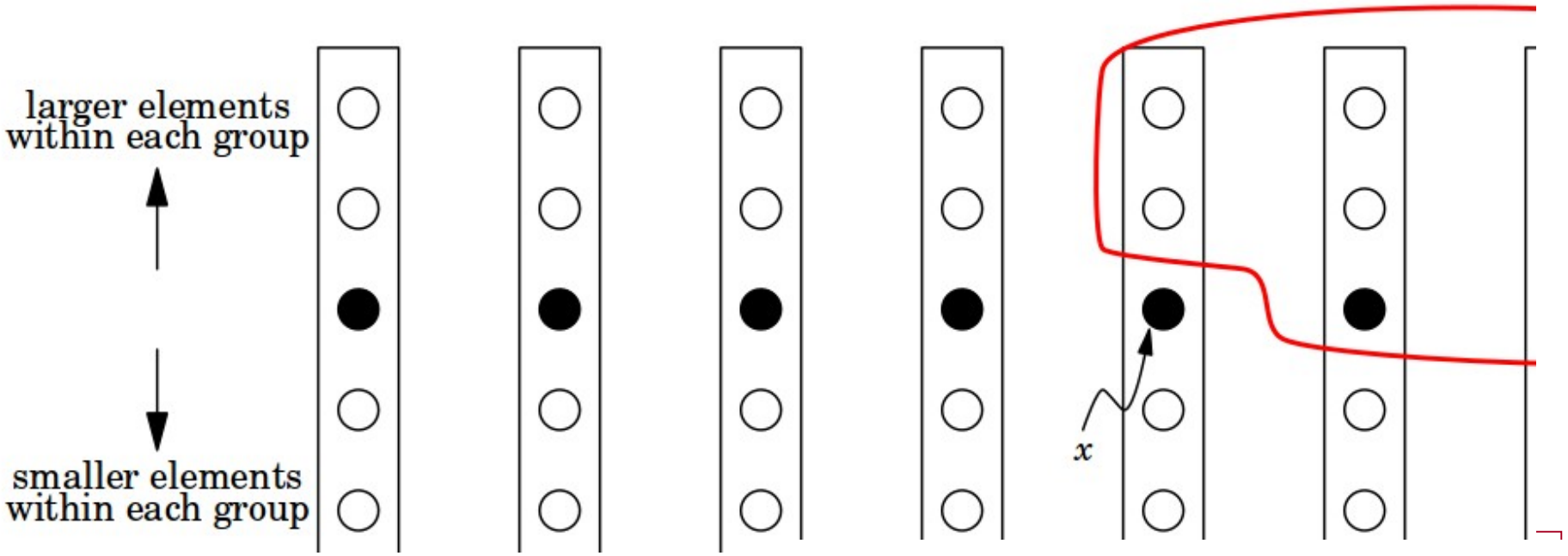
$$= c \cdot N \cdot \log(N) + N \cdot c \cdot (\varepsilon \cdot \log(\varepsilon) + (1-\varepsilon) \cdot \log(1-\varepsilon)) + N$$

3. Sorting

Median Revisited

Function **Split** ($x[]$; l , r ; s)
// splits $x[l..r]$: into $x[l..m]$ with entries $<s$
// and $x[m+1..r]$ those $\geq s$. Returns m.

Input: $N \in \mathbb{N}$ and array $x[1\ldots N]$
W.l.o.g. $x[n] \neq x[m]$ for all $n \neq m$



K-th order statistic: m s.t.
$\#\{n : x_n < x_m\} < K \leq \#\{n : x_n \leq x_m\}$

Lemma: *Median of 5-medians* is "$\leq$" at least $\frac{1}{2} \cdot \frac{3}{5} = 30\%$ of entries, and "$>$" at most 70%.

Output: m such that $0.3 \cdot N \leq \#\{n : x_n \leq x_m\} \leq 0.7 \cdot N + 1$

3. Sorting

Linear-time Median

Function **Partition** ($x[]$; l , r ; y)
// partitions $x[l..r]$ into $x[l..m]$ entries $<y$
// and $x[m+1..r]$ those $\geq y$. Returns m.

Input: $N \in \mathbb{N}$ and array $x[1\ldots N]$
W.l.o.g. $x[n] \neq x[m]$ for all $n \neq m$

Function **OrderStat** ($x[]$; l, r , K)
While $l<r$ do
  Call **ApproxMed** in order to determine an *approximate* median of $x[l\ldots r]$.
  Partition $x[l\ldots r]$ accordingly.
  Proceed to the left (=decrease r) /right (=increase l) accordingly.

Function **ApproxMed** ($x[]$; l , r)
Process $x[l...r]$ in groups of 5:
  For each group, find its median (for instance by brute force).
Then call **OrderStat** to determine the median of these 5medians.

$n := r - l+1$

***K*-th order statistic:** m s.t.
$\#\{n : x_n<x_m\} < K \leq \#\{n : x_n \leq x_m\}$

Lemma: *Median of 5-medians* is "≤" at least ½·⅗ = 30% of entries, and ">" at most 70%.

$$T_A(n) = O(n) + T_O(0.2 \cdot n)\ , \quad T_O(n) = T_A(n) + O(n) + T_O(0.7 \cdot n)$$

3. Sorting

Optimality

Specification: Fix set X with total order $\leq$.

Input: $N \in \mathbb{N}$ and array $x[1 \ldots N]$ with values in X

Output: *Permutation* π s.t. $x_{\pi(1)} \leq \ldots \leq x_{\pi(N)}$

Primitives

"$x[\pi[n]] \leq x[\pi[m]]$?"

Swap $\pi[n] \leftrightarrow \pi[m]$

Index arithmetic

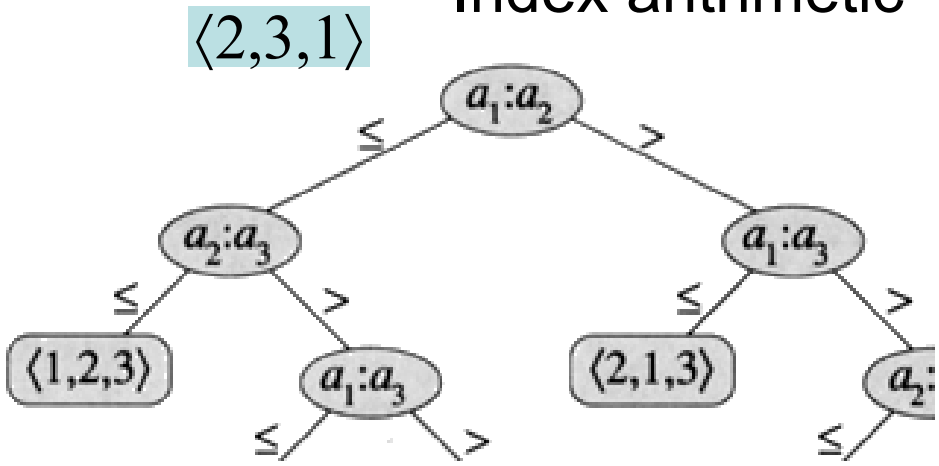
Definition: A *Decision Tree for sorting N elements* is a binary tree

whose nodes are labeled "$x[i] \leq x[j]$?", $1 \leq i < j \leq N$.

and whose leaves are labeled with permutations π such that

every input $x = x[1 \ldots N]$ ends up in a leaf

whose label π satisfies $x[\pi[1]] \leq \ldots \leq x[\pi[N]]$.



3. Sorting

Optimality

Lemma: a) For fixed N, every *comparison-based* sorting algorithm of worst-case runtime $T(N)$ can be "unrolled" into a decision tree for sorting N elements of depth $\leq T(N)$.

b) Every permutation $x=\pi$ ends up in the unique leaf with label π^{-1}.

c) A binary tree with $N!$ leaves has depth $\geq \log(N!)=\Theta(N \cdot \log N)$

Definition: A *Decision Tree for sorting N elements* is a binary tree

whose internal nodes are labeled "$x[i] \leq x[j]$?", $1 \leq i < j \leq N$.

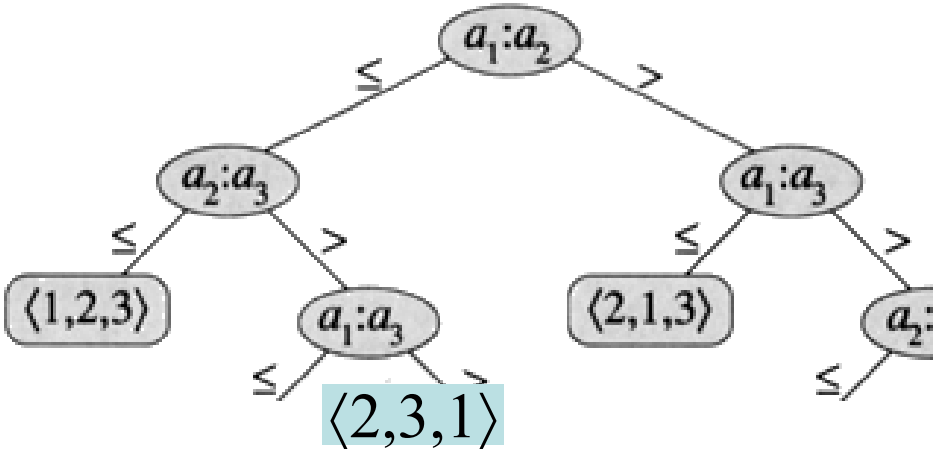
and whose leaves are labeled with permutations π.

Theorem: Any comparison based sorting algorithm requires time at least $\Omega(N \cdot \log N)$.

"$x[\pi[n]] \leq x[\pi[m]]$?"

Swap $\pi[n] \leftrightarrow \pi[m]$

Index arithmetic



3. Sorting

Counting Sort

Specification: Fix set $X=\{1,\ldots M\}$!

Input: $N\in\mathbb{N}$ and array $x[1\ldots N]$ with key values in X

Output: *Permuted* array $y=x\circ\pi$ s.t. $x.key[\pi[1]]\leq\ldots\leq x.key[\pi[N]]$

integer array $count[1\ldots M]$;

For m:=1 to M do $count[m]$:= 0;

For n:=1 to N do $count[x.key[n]]$++;

For m:=2 to M do

$count[m]$ += $count[m-1]$;

Endfor;

For n:=1 to N do

$y[count[x[n].key]$--$]$:= $x[n]$;

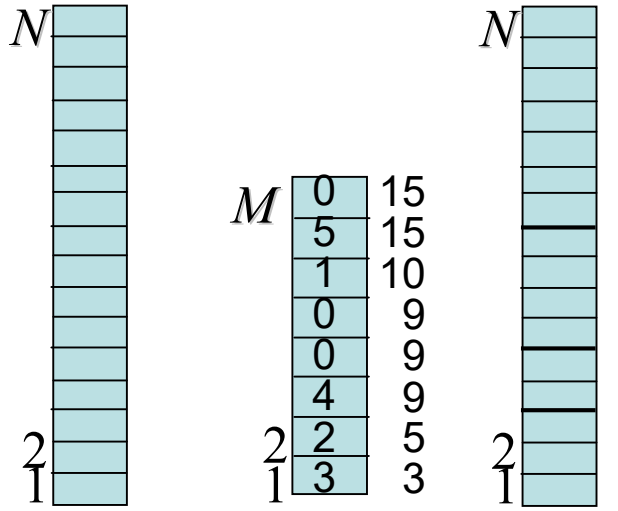
runtime $O(N+M)$

memory $O(N+M)$

~~"$x[\pi[n]]\leq x[\pi[m]]$?"~~

~~Swap $\pi[n]\leftrightarrow\pi[m]$~~

~~Index arithmetic~~



3. Sorting

Radix Sort

Specification: Fix set $X=\{0,1\}^m$ with lexicographical order.

Input: $N\in\mathbb{N}$ and array $x[1\ldots N]$ with values in X

$0\ b_{m-1}\ \ldots\ b_2\ b_1$
$<\ 1\ c_{m-1}\ \ldots\ c_2\ c_1$

Output: *Permuted* array $y=x\circ\pi$ s.t. $y[1]\leq\ldots\leq y[N]$

Quick-/Mergesort
Bit-cost: $O(N\cdot\log N\cdot m)$

"$x[n]\leq x[m]$?"

Swap $x[n]\leftrightarrow x[m]$

Index arithmetic

RadixSort($x[]$, l, r, m);
// Sort $x[l\ldots r]$ w.r.t. bits $\#m\ldots\#1$
If $l=r$ or $m<1$ then return;
// Put all entries with **0** as bit $\#m$ before those with **1**:
mid := **Split** ($x[]$, l , r , m);
RadixSort($x[]$, l , mid , m-1);
RadixSort($x[]$, mid+1 , r , m-1);

$O(N\cdot m)$ operations

Bit-cost: $O(N\cdot m^2)$

3. Sorting *in Parallel*

Sorting Networks

Specification: Fix set X with total order $\leq$.

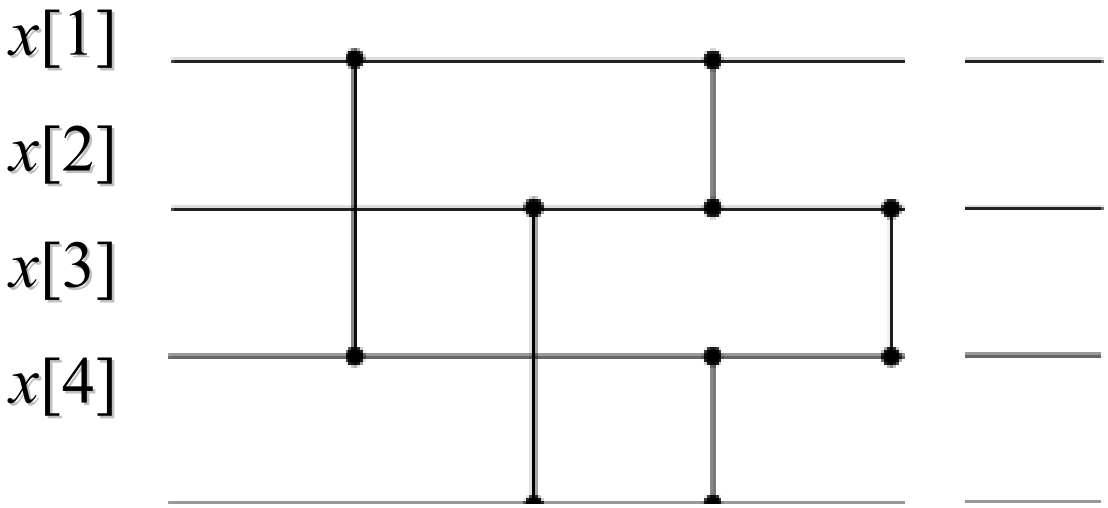
Input: $N \in \mathbb{N}$ and values $x[1 \ldots N]$ in X

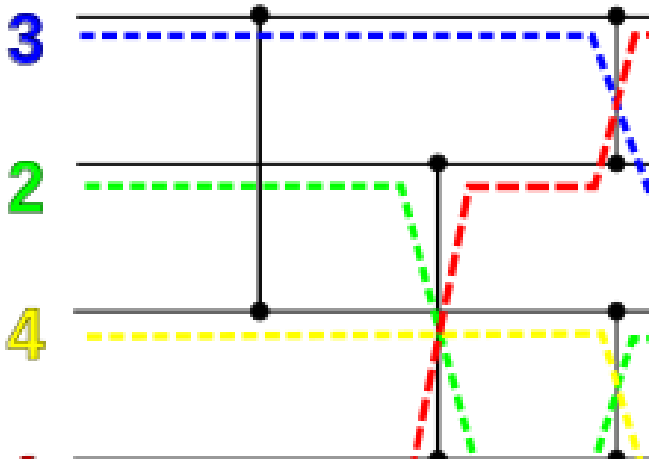
Output: *Permuted* array $y = x \circ \pi$ s.t. $x[\pi[1]] \leq \ldots \leq x[\pi[N]]$

One primitive Gate

"If $x[n] \leq x[m]$ then swap $x[n] \leftrightarrow x[m]$"

N=4:





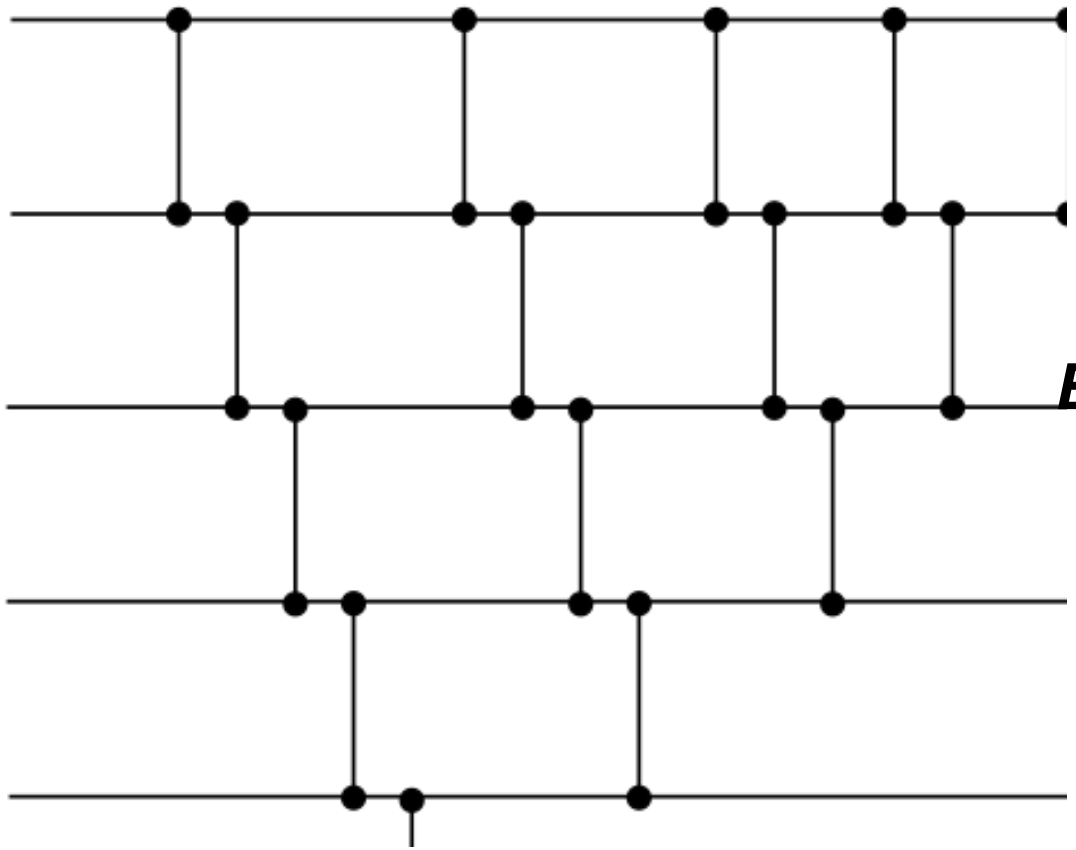
3. Sorting *in Parallel*

Sorting Networks (continued)

Specification: Fix set X with total order $\leq$.

Input: $N \in \mathbb{N}$ and values $x[1 \ldots N]$ in X

Output: *Permuted* array $y = x \circ \pi$ s.t. $x[\pi[1]] \leq \ldots \leq x[\pi[N]]$



One primitive Gate

"If $x[n] \leq x[m]$ then

swap $x[n] \leftrightarrow x[m]$"

***Bubble* Sorting Network:**

Size $O(N^2)$,

Depth (=parallel time) $O(N)$

3. Sorting *in Parallel*

Sorting Networks (continued)

Specification: Fix set X with total order $\leq$.

Input: $N \in \mathbb{N}$ and values $x[1 \ldots N]$ in X

Output: *Permuted* array $y = x \circ \pi$ s.t. $x[\pi[1]] \leq \ldots \leq x[\pi[N]]$

One primitive Gate

"If $x[n] \leq x[m]$ then swap $x[n] \leftrightarrow x[m]$"

Theorem (without proof):

a) If a sorting network correctly sorts all sequences $x[1 \ldots N]$ with **values in {0,1}**,

then it correctly sorts all sequences with **values in X**.

b) There exist sorting networks of size $O(N \cdot \log N)$ and depth $O(\log N)$.

This is optimal

***Bubble* Sorting Network:**

Size $O(N^2)$,

Depth (=parallel time) $O(N)$

Summary

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   - Specification
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   - Sorting in Linear Time
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of Comparison-Based Sorting