Martin Ziegler

Syllabus

6. String Problems

- Recap on Strings
- Pattern Matching: *Knuth-Morris-Pratt*
- Longest Common Substring
- Edit Distance
- Context-free Parsing: *Cocke-Younger-Kasami*
- *Huffman* Compression

6. String Problems

strings recap

Specification: Fix finite alphabet $\Sigma \neq \emptyset$, often {**0**,**1**}

A string over Σ is a finite sequence $s = (s_0, \ldots s_{n-1}) \in \Sigma^*$,

input/output as array $s[0 \ldots n\text{-}1]$.

Terminology: Length $|(s_0,\ldots s_{n-1})| = n$,

concatenation $s \circ t$

prefix = initial segment $(s_0,\ldots s_{n-1})_{<m} = (s_0,\ldots s_{m-1})$ for $m \leq n$.

s prefix of t $\Leftrightarrow$ $\exists u$: $t = s \circ u$

s suffix of t $\Leftrightarrow$ $\exists v$: $t = v \circ s$

s substring of t $\Leftrightarrow$ $\exists u,v$: $t = v \circ s \circ u$

Specification (cont.): Fix finite set $V \neq \emptyset$ disjoint to Σ.

6. String Problems

Pattern Matching

Input: Two strings w and p of lengths $n = |w| \gg |p| = m$.

Output: Does w contain p, and where (first, all) ?

arrays $w[0\ldots n\text{-}1]$ and $p[0\ldots m\text{-}1]$

$w =$ ABCXABCDABXABCDABCDABDE

$p =$ ABCDABD

Naïve algorithm:

For k:=0 to n-1

  If $w[k]=p[0]$ then

    Compare $w[k+1\ldots k+m\text{-}1]$ to $p[1\ldots m\text{-}1]$

    If agree, then output k.

runtime $O(n\cdot m)$

Preprocess pattern:

$T[] =$

-1	0	0	0	-1	0	2	0
	A	B	C	D	A	B	D

6. String Problems — Knuth-Morris-Pratt

Input: Two strings w and p of lengths $n = |w|$ >> $|p| = m$.

Output: Does w contain p, and where (first, all) ?

arrays $w[0 \ldots n\text{-}1]$ and $p[0 \ldots m\text{-}1]$

$w =$ ABCXABCDABXABCDABCDABDE

$p =$ **ABACABABA**

KMP algorithm:

k:=0; j:=0; While $k<n$ do
  If $w[k]=p[j]$ then
    k++; j++;
    If $j=m$ then output $k\text{-}j$; $j:=T[j]$; endif
  else $j:=T[j]$; If $j<0$ then k++; j++; endif

runtime $O(n+m)$

Preprocess pattern:

$T[] =$
-1 0 -1 1 -1 0 -1 3 -1 3
A B A C A B A B A

runtime $O(m)$

6. String Problems

Longest Common Substring

Specification: Fix alphabet Σ

Input: $v \in \Sigma^n$, $w \in \Sigma^m$

Output: Length/positions of longest common substring?

Example: "ABABC" and "BABCA" share "BABC" as longest common substring

Naïve Algorithm:

Try all possible pairs of initial positions

$i=0,\ldots n\text{-}1$ and $j=0,\ldots m\text{-}1.$

For each compare $v[i,..i+k]$ to $w[j,..]$

runtime $O(n \cdot m \cdot \min(n,m))$

6. String Problems

Longest Common Substring

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Specification: Fix alphabet $\Sum$ **Input:** $v \in \Sigma^n$, $w \in \Sigma^m$

Output: Length/positions of longest common substring?

Better Algorithm:

Fill integer table $\mathrm{LCS}[0\ldots n, 0\ldots m]$,
such that $\mathrm{LCS}[i,j]$:= length of
longest common suffix
shared by initial segments
$v[0...i-1]$ and $w[0...j-1]$

runtime $O(n \cdot m)$

$$\mathrm{LCS}[0,j] = 0 = \mathrm{LCS}[i,0]$$

$$\mathrm{LCS}[i+1,j+1] = \mathrm{LCS}[i,j]+1 \quad \text{if } v[i]=w[j]$$
$$= 0 \quad \text{if } v[i]\neq w[j]$$

Example:

	A	B	A	B
B	0	1	0	1
A	1	0	2	0
B	0	2	0	3
A	1	0	3	0

6. String Problems

Edit Distance

Specification: Fix alphabet Σ **Input:** $v \in \Sigma^n$, $w \in \Sigma^m$

Output: Min. # symbol insert/delete op.s converting v into w.

Proposition: This constitutes a metric on Σ^*. runtime $O(n \cdot m)$

Example: "`kitten`" and "`sitting`" have edit distance 5:

`itten`, `sitten`, `sittn`, `sittin`, `sitting`

Wagner-Fischer Algorithm: Fill table $d[0 \ldots n, 0 \ldots m]$
such that $d[i,j]$:= edit distance of $v[0..i\text{-}1]$ and $w[0..j\text{-}1]$

$$d[0,j] = j \qquad d[i+1,j+1] = d[i,j] \qquad \text{if } v[i]=w[j]$$

$$d[i,0] = i \qquad = \min\{\, d[i,j+1]+1 \,,\, d[i+1,j]+1 \,\} \qquad \text{if } v[i] \neq w[j]$$

Variants: Dis/allow (i) replacement, (ii) transposition, (iii) …
Assign positive weights to different operations.

6. String Problems

Grammar

Specification: Fix alphabet Σ, disjoint finite set V of variables and fix a finite set R of rules as well as $S \in V$

Input: $w \in \Sigma^*$. **Output:** Can w be generated from S ?

Example: $V = \{S,X\}$, $\Sigma = \{\mathbf{a},\mathbf{b},\mathbf{c}\}$

three rules $S \to \mathbf{a}X S\mathbf{c}$, $S \to \mathbf{abc}$, $X\mathbf{a} \to \mathbf{a}X$, $X\mathbf{b} \to \mathbf{bb}$

generate precisely the strings $\mathbf{a}^n \mathbf{b}^n \mathbf{c}^n$, $n \in \mathbb{N}$.

Definition: A rule r is an assignment $x \to y$,
where $x,y \in (\Sigma \cup V)^*$ and x contains some variable.

A rule $x \to y$ is context-free, if $x \in V$.

6. String Problems Cocke-Younger-Kasami

Specification: Fix alphabet Σ, disjoint finite set V of variables and fix a finite set R of *context-free* rules as well as $S \in V$

Input: $w \in \Sigma^*$. **Output:** Can w be generated from S ?

Rules in *Chomsky normal form*:

either (i) $X \to YZ$ (one to two variables)

or (ii) $X \to$ **a** (one variable to one symbol)

or (iii) $S \to \varepsilon$ (empty string) [exception only to generate ε..]

Example:
brackets

$S \to$ **(** S **)** S

$S \to \varphi$

Definition: A rule r is an assignment $x \to y$, where $x,y \in (\Sigma \cup V)^*$ and x contains some variable.

A rule $x \to y$ is context-free, if $x \in V$.

6. String Problems Cocke-Younger-Kasami

Rules of the form (i) $X \to YZ$ or (ii) $X \to \mathtt{a}$ or (iii) $S \to \varepsilon$

Table $P[s,l,X]$:= "$w_s, \ldots w_{s+l-1}$ can be generated from variable X".

Input: $w \in \Sigma^n$. **Output:** Can w be generated from S ?

runtime $O(n^3)$

```
Initialize P[..] with false.
For each s = 0 to n-1
  For each rule X → w_s of type (ii)
    P[s,1,X] := true
For l := 2 to n                   // Length of span
  For s := 0 to n-l      // Start of span
    For k := 1 to l-1   // Partition of span
      For each rule of type (i) X → Y Z
        if P[s,k,Y] and P[s+k,l-k,Z]
          then  P[s,l,X] := true
// w can be generated  iff  P[0,n,S] = true.
```

6. String Problems

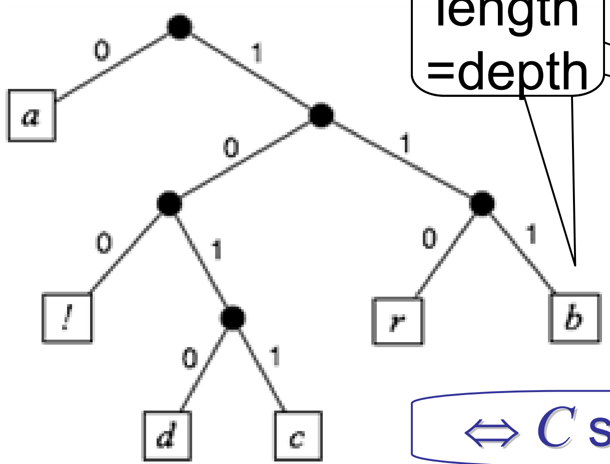
Lossless Compression

Specification: Fix alphabet Σ

Input: $w \in \Sigma^*$ **Output:** "short bit-encoding" of w

1. Determine frequencies f_s of symbols $s \in \Sigma$ in w

"this is an example of a huffman tree"



length =depth

character	encoding
a	0
b	111
c	1011
d	1010
r	110
!	100

⇔ C set of leaves in some bin.tree

Variable length code, need delimiters—or better:

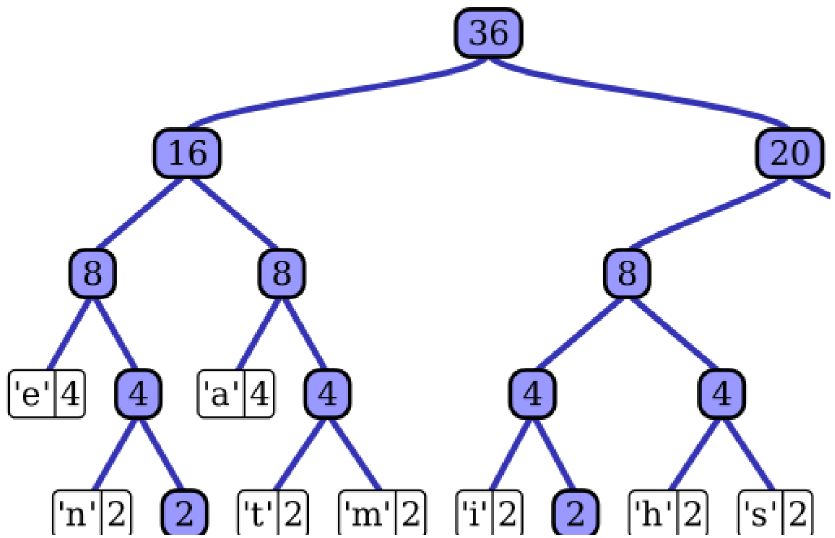
$C \subseteq \Sigma^*$ is *prefix-free* if $v,w \in C$ and $v \triangleleft w \Rightarrow v = w$.

Char	Freq	Co
space	7	11
a	4	01
e	4	00
f	3	11
h	2	10
i	2	10
m	2	01
n	2	00
s	2	10
t	2	01
l	1	11
o	1	00
p	1	10

Specification: Fix alphabet Σ

Input: $w \in \Sigma^*$ **Output:** "short bit-encoding" of w

1. Determine frequencies f_s of symbols $s \in \Sigma$ in w
2. Choose prefix-free $C \subseteq \{0,1\}^*$ / binary tree T
3. Assign to each $s \in \Sigma$ a unique $c_s \in C$ / leaf l_s of T such as to minimize weighted length $\sum_{s\in\Sigma} d(l_s) \cdot f_s$



length =depth

Char	Freq	Co
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l	1	11
o	1	00
p	1	10

6. String Problems

Huffman Tree

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Specification: Fix alphabet Σ

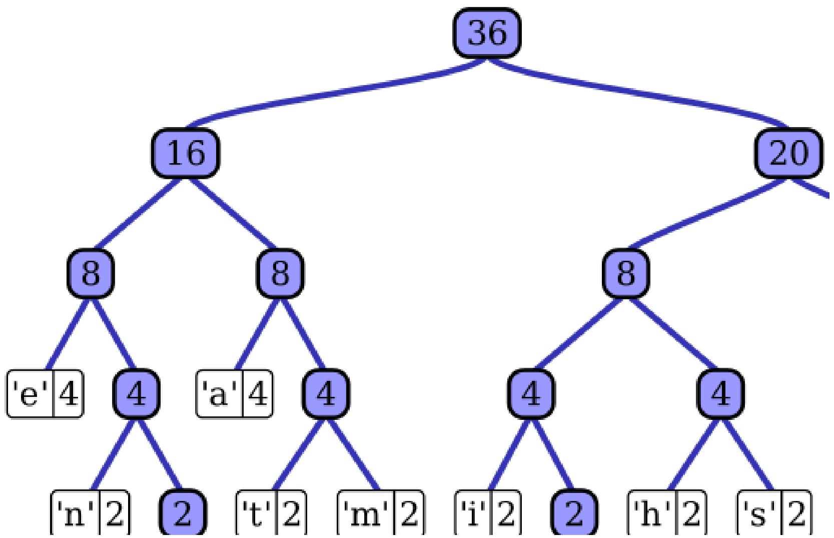
Input: $w \in \Sigma^*$ **Output:** "short bit-encoding" of w

1. Determine frequencies f_s of symbols $s \in \Sigma$ in w

2. Choose prefix-free $C \subseteq \{\mathbf{0,1}\}^*$ / binary tree T

3. Assign to each $s \in \Sigma$ a unique $c_s \in C$ / leaf l_s of T

such as to minimize weighted length $\sum_{s \in \Sigma} \mathrm{d}(l_s) \cdot f_s$



Idea: *Frequent* symbols s (=large f_s) should receive *small* depth $\mathrm{d}(l_s)$,

rare ones can „afford“ large depth

6. String Problems

Huffman Tree

Specification: Fix alphabet Σ

Input: $w \in \Sigma^*$ **Output:** "short bit-encoding" of w

File :

b	p	`	m	j	o	d	a	i	r	u	l	s	e	
1	1	2	2	3	3	3	4	4	5	5	6	6	8	12

Extract two symbols $s,t \in \Sigma^*$ with least frequencies f_s, f_t.

Combine them to a tree with leaves s,t and root $s \circ t$ of frequency $f_{st} := f_s + f_t$.

Repeat.

Idea: *Frequent* symbols s (=large f_s) should receive *small* depth $d(l_s)$, rare ones can „afford" large depth

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Recap

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– Recap on Strings

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– *Huffman* Compression