Martin Ziegler

Syllabus

§8 Randomization

- Motivation: Reliability
- Sources of Randomness
- Las Vegas vs. Monte Carlo
- Primality Testing
- Errors and Amplification
- Blackbox Polynomial Test
- Schwartz-Zippel Lemma
- Perfect Matchings in Graphs
- Matchings via Tutte Determinant

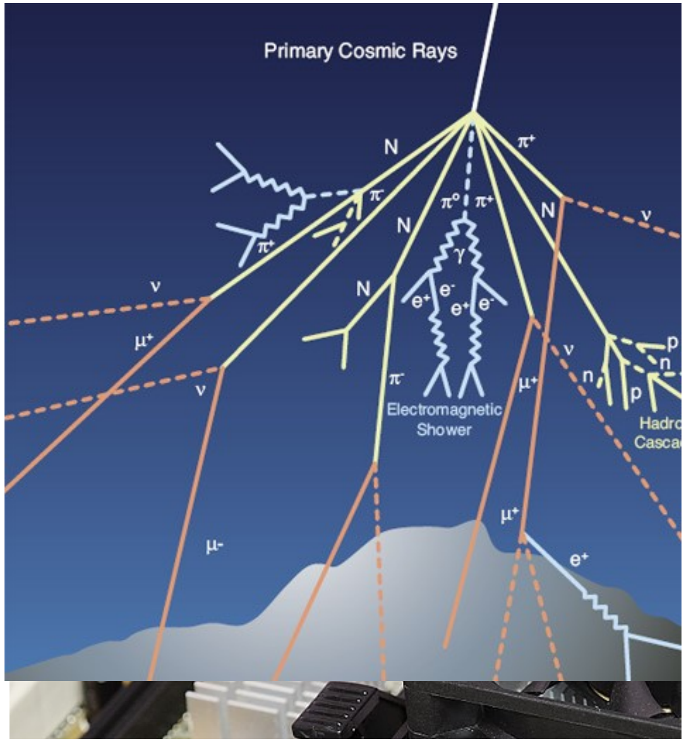
Motivation: *RELIABILITY* ?

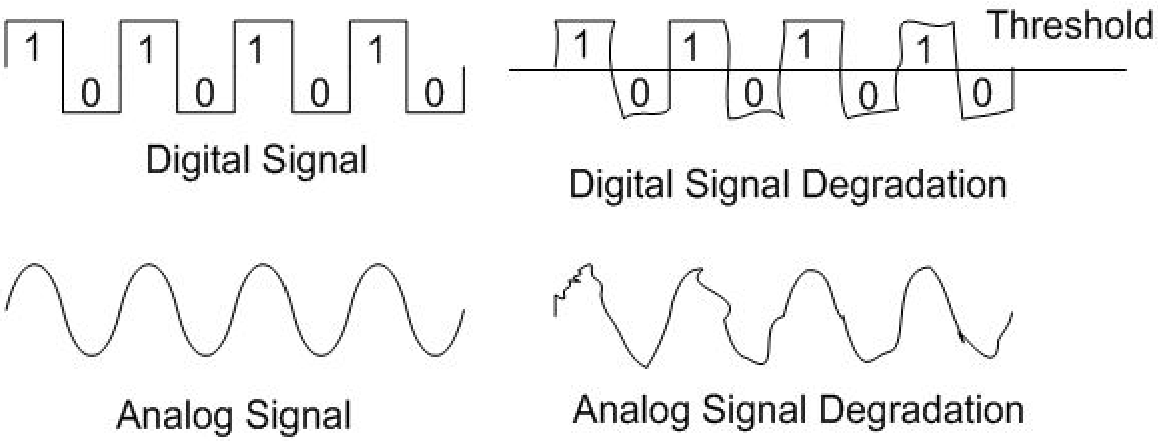
$\approx 10^9$ gates in a PC, flipping $\approx 10^9$ times per second

[Ziegler&Lanford'79] *Effect of Cosmic Rays on Computer Memories*

Table 1: Memory errors per year:

Platf.	Tech.	Per machine				
		CE Incid. (%)	CE Rate Mean	CE Rate C.V.	CE Median Affct.	UE Incid. (%)
A	DDR1	45.4	19,509	3.5	611	0.17
B	DDR1	46.2	23,243	3.4	366	–
C	DDR1	22.3	27,500	17.7	100	2.15
D	DDR2	12.3	20,501	19.0	63	1.21
E	FBD	–	–	–	–	0.27
F	DDR2	26.9	48,621	16.1	25	4.15
Overall	–	32.2	22,696	14.0	277	1.29





1. Efficiency
2. Elegance
3. Reliability: error probability $\leq 2^{-100}$

Sources of Randomness

Entropy from: heat, user, quantum mechanics

• low rate, correlation, bias; (Martin-Löf) Tests

Pseudo-random sequence: deterministic!

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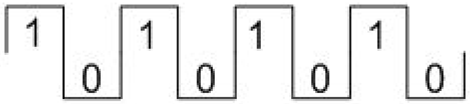
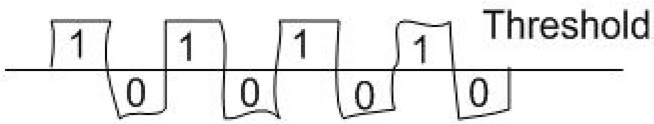
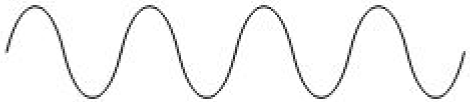


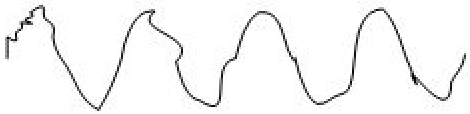
Digital Signal



Digital Signal Degradation



Analog Signal



Analog Signal Degradation

1. Efficiency
2. Elegance
3. Reliability: error probability $\leq 2^{-100}$

Las Vegas vs. Monte Carlo

- Result <u>always</u> correct
- *Expected* runtime

QuickSort?

- Result "*probably*" correct
- <u>Guaranteed</u> runtime

Recap: (Conditional) probability, random variable, (stochastic) independence, expected value, Bayes

Primality Testing

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Decision problem $L \subseteq \mathbb{N}$: Input $x \in \mathbb{N}$, output **`yes/no`**

For $L = \{2,3,5,7,11,\ldots\}$: test **`isprime`**(x) $\quad x \in L \quad x \notin L$

$\boxed{n := \log(x)}$

Naïve idea: *Guess* and verify a proper factor of x.

Case x = product of two primes $\Rightarrow$ detect probability $\approx O(\sqrt{x})$ ↯

Correct-on-average algorithm: Blindly say "*composite*" ☺

Miller-Rabin Test: *Guess* $a<x$. Write $x-1=d\cdot 2^e$ with d odd.

If (*) holds, then say "x is *composite*".

(*) $\quad a^d \not\equiv 1 \pmod{x}$ and $a^{d\cdot 2^r} \not\equiv -1 \pmod{x}$ for all $r<e$,

Theorem: a) If (*) holds, then x is composite. amplify!

b) If x is composite, then $\geq$¼ of all $a<x$ satisfy (*).

Deterministic test [Agrawal/Kayal/Saxena'02]: time $O(n^7)$.

Errors and Amplification

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Decision problem $L \subseteq \mathbb{N}$: Input $x \in \mathbb{N}$, output **yes/no**

$x \in L$ $x \notin L$

Algorithm $\mathcal{A}$ with one-sided error:

- false positives *only*: **yes** but $x \notin L$
- false negatives *only*: **no** but $x \in L$

Error probability
$p^k \ll 1$

$\mathcal{A}'$: Repeat $\mathcal{A}$ k-times, report **no** (only) if *all* return **no**.

Example: $p=0.99$, $k=70 \cdot 100$

Algorithm $\mathcal{B}$ with two-sided error:

both false positive and false negative

Error probability
$p<½$

$\mathcal{B}'$: Repeat $\mathcal{B}$ k-times and report the majority answer.

Example: $p=1/3$, $k=100$

Fact (Hoeffding): Let $X_1,..X_k$ independent random variables in $[0;1]$, $\underline{X} := (X_1+\ldots+X_k)/k$ and $t>0$.
Then $\mathbb{P}[\ \underline{X} \geq \mathbb{E}[\underline{X}] + t\ ] \leq e^{-2kt^2}$

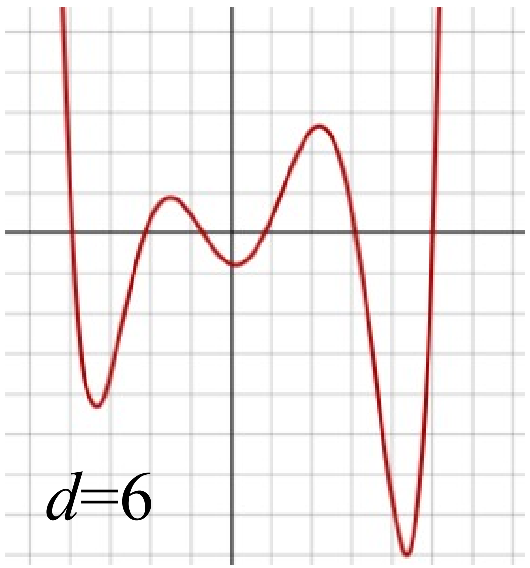
$X_n := n$-th execution errs, $t:=½-p$

Blackbox Polynomial Test

$\underline{x}$ → Fixed unknown polynomial P $\equiv 0$? → $P(\underline{x})$ $=0$?

promise: $\deg(P) \leq d$

Recap: A non-zero *uni*variate polynomial of $\deg \leq d$ has at most d roots. *Total degree* (e.g. of $X^2 \cdot Y^3$ is 5).



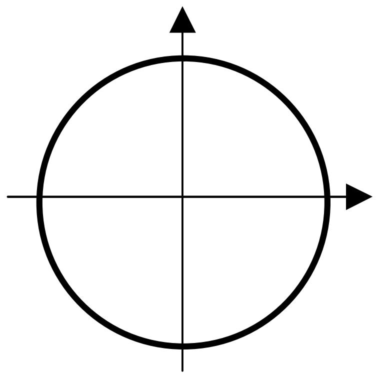
<u>Generic blackbox polynomial test P:</u>

Fix finite set D.

Sample $\underline{x} \in D$ uniformly at random.

If $P(x)=0$ say "$P\equiv 0$".

*multi*variate?



X^2+Y^2-1

Schwartz-Zippel Lemma

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promise: $\deg(P) \le d$

$\underline{x} \longrightarrow$ Fixed unknown polynomial $P \equiv 0$? $\longrightarrow P(\underline{x})$ $=0$?

[*Schwartz'80, Zippel'79, deMillo&Lipton'78*)**:** Fix finite $S \subseteq \mathbb{C}^n$.

Suppose $0 \not\equiv P \in \mathbb{C}[X_1,\ldots X_n]$ has total degree $\le d$.

Sample $x_1,\ldots x_n$ from S independently uniformly at random.

Then $\mathbb{P}\,[\, P(x_1,\ldots x_n)=0 \,] \;\le\; d/|S|$.

$|S|:=2d$, *then* amplify!

Proof: Write $0 \not\equiv P(X_1,\ldots X_n) = \sum_{0\le j\le d} P_j(X_1,\ldots X_{n-1}) \cdot X_n^{\,j}$.

Let j be maximal s.t. $P_j \not\equiv 0$. Then $\mathbb{P}\,[\, P(x_1,\ldots x_n)=0 \,] \;\le$

$$\le \underbrace{\mathbb{P}[\, P_j(x_1,\ldots x_{n-1})=0]}_{\le (d-j)\,/\,|S|} + \underbrace{\mathbb{P}[\, P(x_1,\ldots x_n)=0 \mid P_j(x_1,\ldots x_{n-1}) \neq 0]}_{\le j\,/\,|S|}$$

$$\boxed{\mathbb{P}\,[A] \;=\; \mathbb{P}\,[A \wedge B] + \mathbb{P}\,[A \wedge \neg B] \;\le\; \mathbb{P}\,[B] + \mathbb{P}\,[A|\neg B] \cdot \cancel{\mathbb{P}\,[\neg B]}}$$

Graphs with Perfect Matchings

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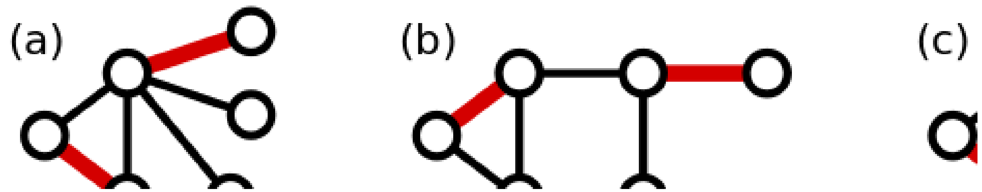
$G=(V,E)$ undirected simple graph.

Does it have a perfect matching?



$t_{ij} := +X_{ij}$ if $\{i,j\} \in E$ and $i<j$
$t_{ij} := -X_{ji}$ if $\{i,j\} \in E$ and $i>j$
$t_{ij} := \; 0$ otherwise.

Tutte matrix $T_G=(t_{ij})$, where X_{ij} are variables, $1 \le i < j \le n = |V|$

[Edmonds'65] Deterministic polynom.-time *Blossom Algorithm*.

- $\det(T_G)$ is an n^2-variate integer polynomial of total degree $\le n$
- Can *evaluate* $\det(T_G)$ using $O(n^3)$ tests&arithmetic operations

Gauss

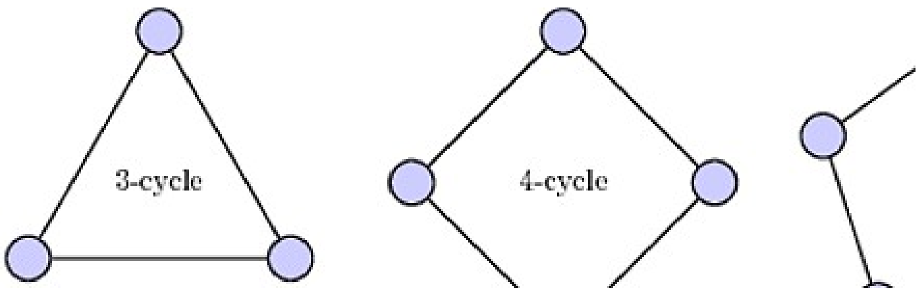
Theorem: $\det(T_G) \equiv 0$ iff G has *no* perfect matching.

Leibniz: $\det(T_G) = \sum_\pi \operatorname{sgn}(\pi) \cdot t_{1,\pi(1)} \cdot t_{2,\pi(2)} \cdot t_{3,\pi(3)} \cdot t_{3,\pi(3)} \cdots t_{n,\pi(n)}$

where $\operatorname{sgn}(\pi)$ = parity of #*odd* cycles in cycle decomposition

A perfect matching of $G=(V,E)$ is a permutation $\pi: V \to V$ s.t.:
$\forall i$: $\{i,\pi(i)\} \in E$ (*) and all cycles have length 2.

Matchings via Tutte Determinant



$t_{ij} := +X_{ij}$ if $\{i,j\} \in E$ and $i<j$
$t_{ij} := -X_{ji}$ if $\{i,j\} \in E$ and $i>j$
$t_{ij} :=\ 0$ otherwise.

Lemma 1: If G has a perfect matching, then $\det(T_G) \not\equiv 0$.

Lemma 2: If $M_\pi \not\equiv 0$ for π composed from *even* cycles only, then G has a perfect matching.

Lemma 3: $M_{\pi'} = -M_\pi$ if π' is π with one *odd* cycle reversed.

Theorem: $\det(T_G) \equiv 0$ iff G has *no* perfect matching.

Leibniz: $\det(T_G) = \sum_\pi \overbrace{\mathrm{sgn}(\pi) \cdot t_{1,\pi(1)} \cdot (\pm X_{ij}) \cdot t_{3,\pi(3)} \cdot (\mp X_{ij}) \cdots t_{n,\pi(n)}}^{=:M_\pi}$

where $\mathrm{sgn}(\pi)$ = parity of #*odd* cycles in cycle decomposition

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