CS204 Discrete Mathematics, Spring 2018

Recitation #2

Time: 2018.3.15 (Thu) 19:00 ~ 20:00

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Suppose the domain of discourse is the set of positive integers $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$. In this recitation, our goal is to prove the following theorem:

Theorem 1. If n is prime and $n \geq 6$, then n is not triangular.

We will work with these definitions of prime numbers and triangular numbers:

Definition 2. n is prime if and only if $n \geq 2$ and there do not exist $a, b \geq 2$ such that $n = ab$.
Definition 3. n is triangular if and only if there exists k such that $n = 1 + 2 + 3 + \cdots + k$.

We will make use of the following lemma:

Lemma 4. n is triangular if and only if there exists k such that $n = \frac{k(k+1)}{2}$.

<u>**Problem 1.**</u> Let $P(n)$ be the proposition "n is prime" and $T(n)$ be the proposition "n is triangular". Express Theorem 1 and Definition 2 in first-order logic. Don't forget to include "for all n" at the beginning of each statement; they are usually implied when writing down in words.

Solution.

Theorem 1: $\forall n \left(\left(P(n) \land (n \geq 6) \right) \to \neg T(n) \right)$

Definition 2: $\forall n \left(P(n) \leftrightarrow \left((n \geq 2) \land \neg\left(\exists a \exists b \; (n = ab) \right) \right) \right)$

<u>**Problem 2.**</u> We will prove Lemma 4 by induction.

a) What is proof by induction? What do we need in a proof by induction?
b) Devise a proposition $P(k)$ to be proved by induction.
c) Show how $P(k)$ helps us to prove Lemma 4.
d) Prove the basis step $P(1)$.
e) Prove the inductive step $P(k) \to P(k+1)$.

Solution.

a) Check textbook. We need a proposition, a proof of the basis step, and a proof of the inductive step.
b) $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$
c) Here, $P(k)$ helps us to prove Lemma 4 because we can replace $1 + 2 + 3 + \cdots + k$ in Definition 3 with $\frac{k(k+1)}{2}$. The rest of the statement is the same, and since Definition 3 is an axiom (and hence true), Lemma 4 will also be true.
d) The left hand side is simply 1. The right hand side is $\frac{1 \cdot 2}{2} = 1$.

e) Here the first equation uses the inductive hypothesis $P(k)$: $1+2+3+\cdots+k=\frac{k(k+1)}{2}$.

$$\begin{aligned}1+2+3+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1)\\ &= (k+1)\left(\frac{k}{2}+1\right)\\ &= (k+1)\left(\frac{k+2}{2}\right)\\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

<u>Problem 3.</u> We will begin the proof of Theorem 1 by making it easier to prove.

a) What is proof by contraposition?
b) We want to prove Theorem 1 using contraposition. What will be our assumption?
c) And what do we need to prove?
d) Why do we want to use proof by contraposition here?

Solution.

a) Check textbook.
b) n is triangular.
c) It is not the case that n is prime and $n \geq 6$. In other words, n is not prime or $n < 6$.
d) The assumption n is prime is actually difficult to work with. Look at the definition (Definition 2): it says there "do not exist a, b". It's often much easier to prove that something exists, since we can then prove it by giving a construction for it.

<u>Problem 4.</u> We will continue the proof of Theorem 1 by solving the easy parts.

a) What is proof by cases?
b) An option is to divide into two cases: $n < 6$ and $n \geq 6$. Why do we want to do this?

Solution.

a) Check textbook.
b) Consider the contraposition obtained from Problem 3. The conclusion reads "n is not prime or $n < 6$". One of the cases immediately completes this conclusion: if "$n < 6$" is true then the disjunction "n is not prime or $n < 6$" is also true. So by dividing into this case, we only need to prove when $n \geq 6$, and we're hoping that we can show n is not prime.

<u>Problem 5.</u> We will finish the rest of the proof.

a) Prove that if $\frac{k(k+1)}{2} \geq 6$ then $k \geq 3$. (Hint: Algebra.)
b) Prove that one of k and $k+1$ is even (and hence divisible by 2). (Hint: Cases.)
c) Complete the proof by combining all the above problems.
d) What is proof by contradiction? Can we use proof by contradiction here?
e) Prove that the number 8 is not prime and not triangular. Is this a counterexample? Why?
f) Prove that the number 3 is prime and triangular. Is this a counterexample? Why?

Solution.

a) Solve the quadratic inequality. You should get $(k+4)(k-3) \geq 0$; since k is a positive integer, in particular $k > 0$, so $k \geq 3$.
b) Divide into cases: when k is even and when k is odd. When k is even there's nothing to prove. When k is odd then $k+1$ is even (assume $k = 2m+1$, then $k+1 = 2(m+1)$).
c)

We will prove the contraposition: if n is triangular then n is not prime or $n < 6$. (Problem 3)

We now divide into two cases: $n < 6$ and $n \geq 6$. If $n < 6$, the conclusion immediately follows. (Problem 4) Thus we may now assume $n \geq 6$ and try to show n is not prime.

By Lemma 4, since n is triangular, there exists k such that $n = \frac{k(k+1)}{2}$. Since $n \geq 6$, we have $k \geq 3$. (Problem 5a) We now divide into two cases again: k is even and k is odd.

Case 1: k is even

Let $k = 2m$. Since $k \geq 3$, it follows $m \geq 1.5$; since m is an integer, $m \geq 2$. This means we can express $n = m \cdot (k+1)$. But both m and $k+1$ are ≥ 2; thus by Definition 2, n is not prime.

Case 2: k is odd

Let $k = 2m+1$. Since $k \geq 3$, it follows $m \geq 1$. Also, $k+1 = 2m+2 = 2(m+1)$. This means we have $n = (k+1) \cdot (m+1)$. But both $k+1$ and $m+1$ are ≥ 2; thus by Definition 2, n is not prime.

In both cases, we obtain that n is not prime. This concludes the proof. ∎

d) Check textbook. We can use proof by contradiction here: the negation of the theorem is "there exists n such that n is prime, $n \geq 6$, and n is triangular". Then we approach like above: let $n = \frac{k(k+1)}{2}$ (because n is triangular), go to Case 2 of the above (because $n \geq 6$), and derive a contradiction (because n should be prime).
e) 8 is not prime ($a = 2,\ b = 4$). 8 is not triangular ($\frac{k(k+1)}{2} = 8$ has non-integral solutions for k). This is not a counterexample since the theorem is in the form $F \to T$; a false premise, or a true conclusion, doesn't make the theorem wrong.
f) 3 is prime (if there were a, b, we would have $n = ab \geq 2 \cdot 2 = 4$). 3 is triangular ($k = 2$). This is not a counterexample since $3 < 6$, so the premise is false.

__Problem 6*.__ Prove that every integer ≥ 5 is the sum of a triangular number of primes, possibly with repetition. For example, $13 = 3 + 5 + 5$ is the sum of three primes (3, 5, 5), and 3 is a triangular number. (Hint: Proof by induction with base case 20 with a separate claim when you reach numbers in the form $k(k + 1)$, and proof by cases for below 20.)

Solution.

This is intended to be a difficult problem for students that are interested in additional challenge. All steps can be followed with what we already knew, but *finding* the steps is the hard part.

Let the integer be n. We divide into two cases: $5 \leq n \leq 19$ and $n \geq 20$. The first case itself is actually completely casework; you have to find a solution for each n. Sample solutions:

- $5 = 5$
- $6 = 2 + 2 + 2$
- $7 = 2 + 2 + 3$
- $8 = 2 + 3 + 3$
- $9 = 3 + 3 + 3$
- $10 = 2 + 3 + 5$
- $11 = 11$
- $12 = 2 + 2 + 2 + 2 + 2 + 2$
- $13 = 2 + 2 + 2 + 2 + 2 + 3$
- $14 = 2 + 2 + 2 + 2 + 3 + 3$
- $15 = 2 + 2 + 2 + 3 + 3 + 3$
- $16 = 2 + 2 + 3 + 3 + 3 + 3$
- $17 = 2 + 3 + 3 + 3 + 3 + 3$
- $18 = 3 + 3 + 3 + 3 + 3 + 3$
- $19 = 19$

We can see two patterns (6-9 and 12-18) there. The second case of $n \geq 20$ will actually exploit this very pattern.

Lemma 6. Assume $n \geq 20$; choose k such that $k(k + 1) \leq n < (k + 1)(k + 2)$. Then we can write n as the sum of $n - k(k + 1)$ instances of 3 and $\frac{3}{2} \cdot k(k + 1) - n$ instances of 2. In particular, we can represent n as the sum of a triangular number of primes.

We can prove this lemma directly:

- 2 and 3 are both primes.
- The number of terms $\big(n - k(k + 1)\big) + \left(\frac{3}{2} \cdot k(k + 1) - n\right) = \frac{k(k+1)}{2}$ is a triangular number.
- The sum $3\big(n - k(k + 1)\big) + 2\left(\frac{3}{2} \cdot k(k + 1) - n\right) = n$ is indeed n.

Or we can prove it by induction, which is likely how one would approach this problem. Let $P(n)$ be the statement of Lemma 6. Denote $S(n)$ to be the representation of n as given in the statement of the lemma; for example, $S(23) = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 3 + 3 + 3$, since $n = 23$ gives the value $k = 4$, an substituting these to the counts, we need $n - k(k + 1) = 23 - 20 = 3$ instances of 3 and $\frac{3}{2} \cdot k(k + 1) - n = 30 - 23 = 7$ instances of 2.

The basis step is $P(20)$, where we can represent 20 as the sum of ten 2's ($k = 4$). Plugging this into the counts, we have $k(k + 1) = 20$, so there are $n - k(k + 1) = 20 - 20 = 0$ instances of 3 and $\frac{3}{2} \cdot k(k + 1) - n = 30 - 20 = 10$ instances of 2, matching our claim.

The inductive step is $P(n) \to P(n+1)$, and is divided into two cases:

- If $n+1 = k(k+1)$ for some k, then we simply represent $n+1$ as the sum of $\frac{k(k+1)}{2}$ instances of 2. (This is a trivial proof, where we don't use the assumption $P(n)$ at all.) Note that the counts are once again accurate: there are $(n+1) - k(k+1) = 0$ instances of 3 and $\frac{3}{2} \cdot k(k+1) - (n+1) = \frac{k(k+1)}{2}$ instances of 2.
- Otherwise, we take the representation of n, and change a 2 into a 3. This increases the sum by 1 and doesn't change the number of terms. If we had $n - k(k+1)$ instances of 3 in $S(n)$, we now have $(n+1) - k(k+1)$ instances of 3 in $S(n+1)$; likewise, if we had $\frac{3}{2} \cdot k(k+1) - n$ instances of 2 in $S(n)$, we now have $\frac{3}{2} \cdot k(k+1) - (n+1)$ instances of 2 in $S(n+1)$.

The only way the second case can fail is if there is no 2 in $S(n)$. But note that by inductive hypothesis, there are $\frac{3}{2} \cdot k(k+1) - n$ instances of 2 in $S(n)$. Since we assume $n < (k+1)(k+2)$, this means:

$$\begin{aligned}
\frac{3}{2} \cdot k(k+1) - n &> \frac{3}{2} \cdot k(k+1) - (k+1)(k+2) \\
&= (k+1)\left(\frac{3}{2}k - k - 2\right) \\
&= (k+1)\left(\frac{k-4}{2}\right) \\
&\geq 0
\end{aligned}$$

(The last inequality uses the fact that if $n \geq 20$, then $k \geq 4$.) This means there is at least a 2 in $S(n)$, and so the second case will not fail; that is, we can always perform the inductive step.

This completes the inductive step, and hence the entire proof. ■