„Introduction to Algorithms"

Syllabus

4. Graph Problems

- Recap on Graphs: un/directed, weighted
- Shortest Paths: single-source, all-pairs
- Minimum Spanning Tree: Prim, Kruskal
- Maximum Flow: Ford-Fulkerson, Edmonds-Karp
- Maximum (weighted) Bipartite Matching
- Minimum Cut

4. Graph Problems

graph recap

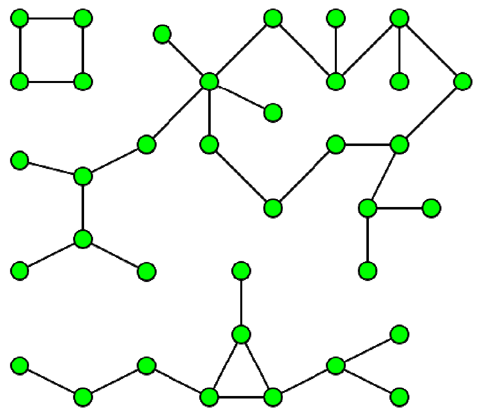
Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Basic graph concepts:
- simple: no *multi*-edges nor loops
- in-/out-/degree
- (un-/directed) path
- (strongly) connected component
- subgraph, induced graph

$E \subseteq V\times V$ directed edges

E symmetric: *un*directed edges

$w:E\to\mathbb{N}$ edge weights ∞/0: absent, 1: present



Handshaking lemma:

$$\#E = \sum_{v\in V} \mathrm{indeg}(v) = \sum_{v\in V} \mathrm{outdeg}(v)$$

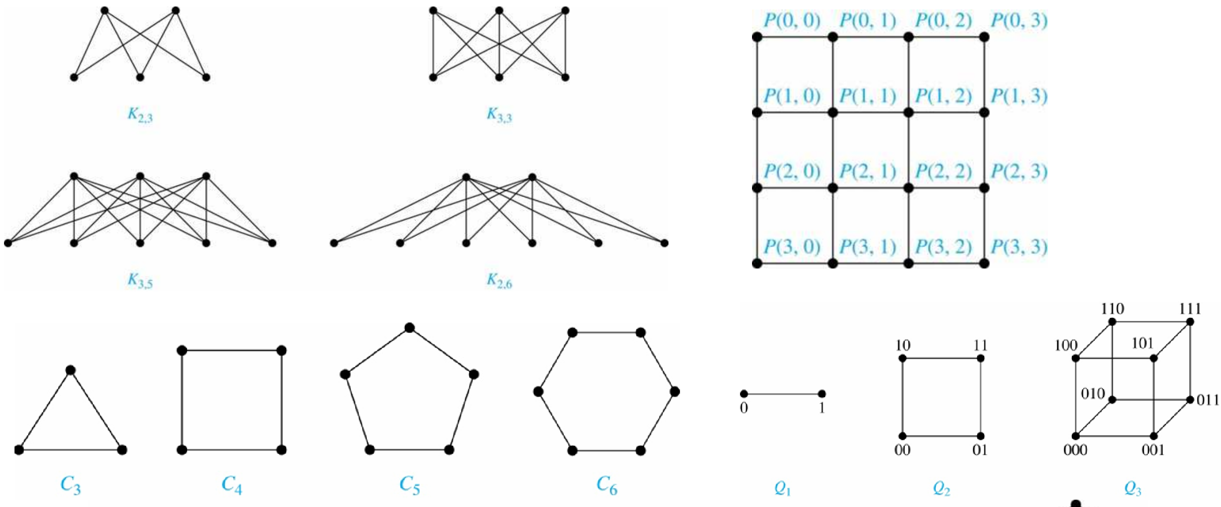
Adjacency/weight matrix

$A_G \in \mathbb{N}^{V\times V}$ Powers of A_G

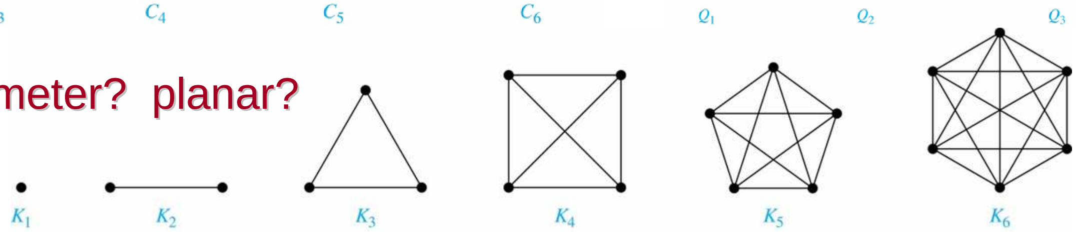
4. Graph Problems

graph examples

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges



diameter? planar?



4. Graph Problems

Connectedness

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: A_G; $s,t\in V$

Adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: Is there a (directed) path from s to t in G?

$A_{u,v} = \infty$ *no* edge

$A_{u,u} = 0$

DFS(v) // Is t reachable in G from v?

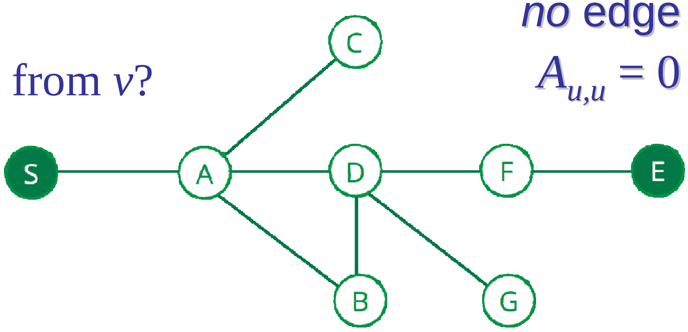
If v is marked *visited*
  Return (**false**);

If $v=t$ Return (**true**);

Mark v as *visited*;

For each neighbor u of v do
  if DFS(u) Return (**true**);

Return (**false**);



Reachable(G,s,t)

For each vertex $v\in V$
  Mark v as *unvisited;*

Return *DFS*(s)

4. Graph Problems

Shortest path(s)

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: A_G; $s\in V$ Adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: For every $t\in V$, weight $d(s,t)$ of lightest path from s to t.

Dijkstra's Algorithm:

Mark all vertices *un*visited. (set of *un*visited vertices)

Initialize $Q:=V$.

For each vertex v let $d_v:=\infty$; $d_s:=0$. (tentative distance from s)

While $Q \neq \emptyset$ do — Correctness???

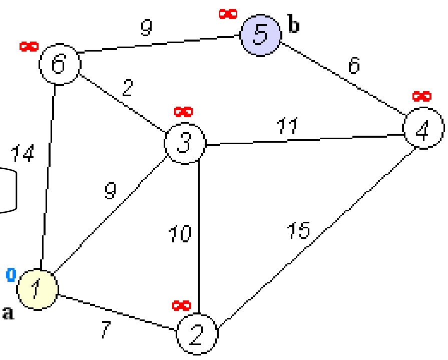
Extract from Q a vertex u with least d_u. Mark u as *visited*.

For each *un*visited neighbor u of v do

If $d':=d_u+A_{uv} < d_v$ then decrease $d_v:=d'$.

$O(n\cdot extractMin+m\cdot decreaseKey)$

array $O(n\cdot n+m\cdot 1)$



4. Graph Problems

All shortest paths

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: A_G Adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: For every $t\in V$, weight $d(s,t)$ of lightest path from s to t.

Loop invariant $d_v \geq d(s,v)$. Suppose $M := \{\, v : d_v > d(s,v) \,\} \neq \emptyset$.
Then $\delta := \min\{\, d(s,v) : v\in M \,\}$ and $v\in M$ with $d(s,v)=\delta$ exist.
For $(s, \dots, u, v)$ a lightest path to v, it holds $\delta > d(s,u) = d_u$.
Thus $d(s,v) = d(s,u)+A_{uv}$ and u gets extracted from Q before v. ↯

For correctness, recall main loop: While $Q \neq \emptyset$ do

Extract from Q a vertex u with least d_u. Mark u as *visited*.

For each *un*visited neighbor u of v do — in increasing order w.r.t. d

If $d':=d_u+A_{uv} < d_v$ then decrease $d_v:=d'$.

4. Graph Problems — *All* shortest paths

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: A_G — Adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: For all $s,t\in V$, <u>weight</u> $d(s,t)$ of lightest path from s to t.

$A_{u,v} = \infty$ *no* edge

$A_{u,u} = 0$

Floyd-Warshall Algorithm:

For all pairs (u,v) of vertices, initialize $d_{u,v}:=A_{u,v}$

For each vertex $u\in V$

  For each vertex $v\in V$

    For each vertex $w\in V$

      If $d_{v,w} > d_{v,u} + d_{u,w}$ then

        $d_{v,w} := d_{v,u} + d_{u,w}$

runtime $O(n^3)$

Correctness?

4. Graph Problems — *Min.* Spanning Tree

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

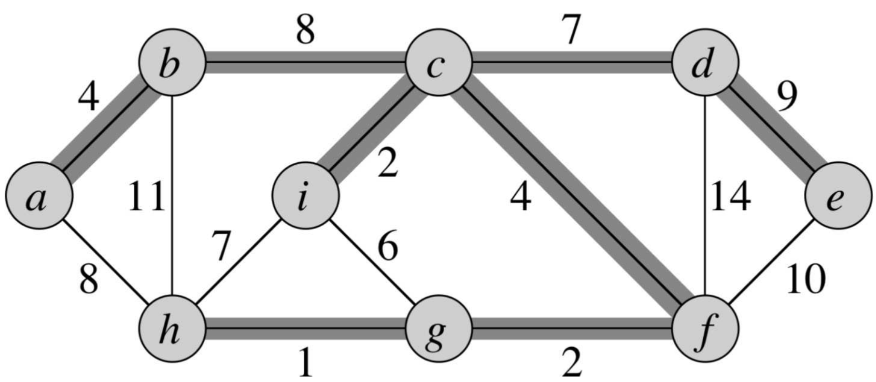
Input: A_G — **Symmetric** adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: $T\subseteq E$ <u>spanning tree</u> of least weight

s.t. (V,T) connected

$A_{u,v} = \infty$ *no* edge

$A_{u,u} = 0$



4. Graph Problems

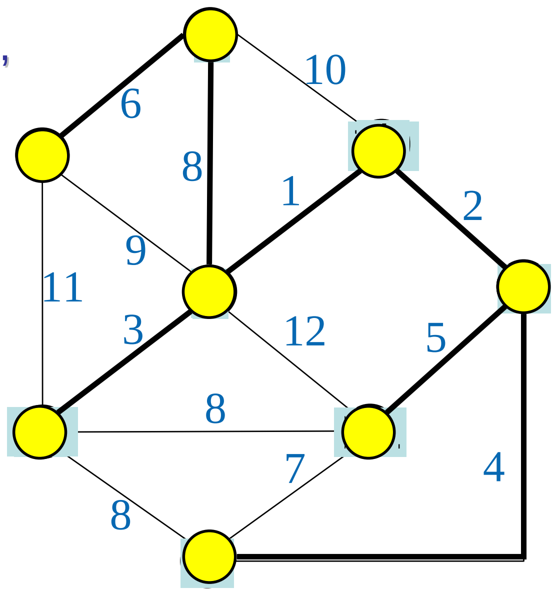
Prim's Algorithm

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: A_G **Symmetric** adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: $T\subseteq E$ spanning tree of least weight

1. Initialize a tree with a single vertex, chosen arbitrarily from the graph.
2. Grow the tree by one edge: Of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and transfer it to the tree.
3. Repeat step 2 (until all vertices are in the tree).



4. Graph Problems

Prim's Algorithm

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: A_G **Symmetric** adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: $T\subseteq E$ spanning tree of least weight

$O(n\cdot extractMin + m\cdot decreaseKey)$

Initialize $F:=\emptyset$, $Q:=V$. Also: $d_v:=\infty$ and $e_v:=0$ for all $v\in V$.

While $Q\neq\emptyset$ do

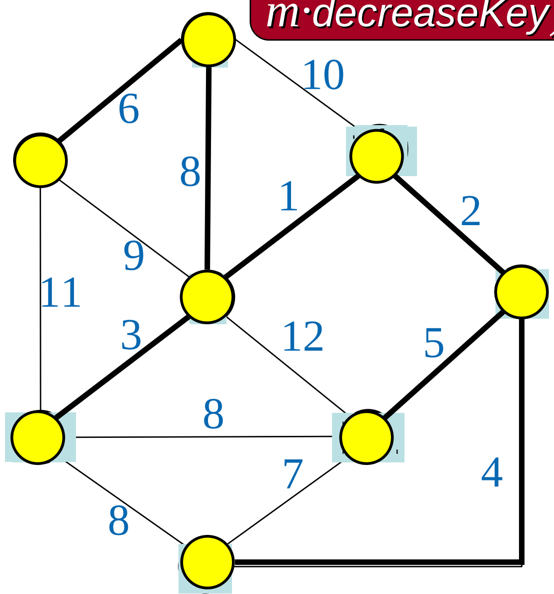
Extract from Q a vertex u with least d_u.

If $e_u\neq 0$, add edge (u,e_u) to F.

For each neighbor $v\in Q$ of u do

If $A_{uv} < d_v$ then

decrease $d_v := A_{uv}$; $e_v:=u$;



4. Graph Problems

Kruskal Algorithm

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: A_G **Symmetric** adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: $T\subseteq E$ spanning tree of least weight

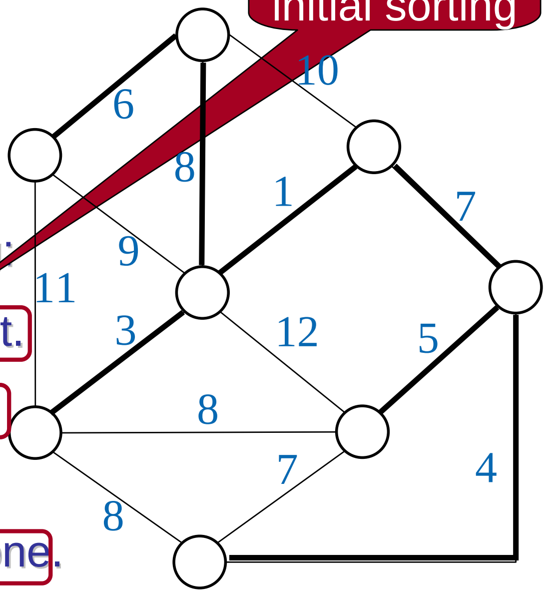
O(1) after initial sorting

Initialize the forest (=set of trees) with edges F:={}, i.e., such that each vertex $v\in V$ is a separate tree.

While $E\neq${} and F is not yet spanning:

- Extract from E edge e of least weight.
- If e connects two different trees of F then add e to F, thus combining two trees into a single one.

?



4. Graph Problems

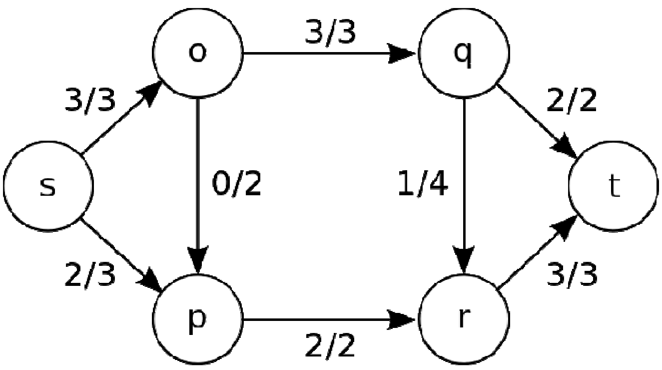
Max Flow

Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: $s,t\in V$, A_G adjacency/weight matrix $A_G \in \boxed{\mathbb{N}}^{V\times V}$

Output: $f:E\to\boxed{\mathbb{N}}$ max. flow from s to t

Goal: maximize $\sum_{v:(s,v)\in E} f(s,v)$
$= \sum_{u:(u,t)\in E} f(u,t)$
f flow (from s to t)



Lemma: There exists an *integral* maximal flow.

Def: A flow from s to t in G with weights $A\geq 0$ is a function $f:E\to\mathbb{R}$ such that $\forall v\in V\setminus\{s,t\}$: $\sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w)$.
It is admissible if it holds $f(u,v)\leq A_{u,v}$

4. Graph Problems *Ford-Fulkerson*

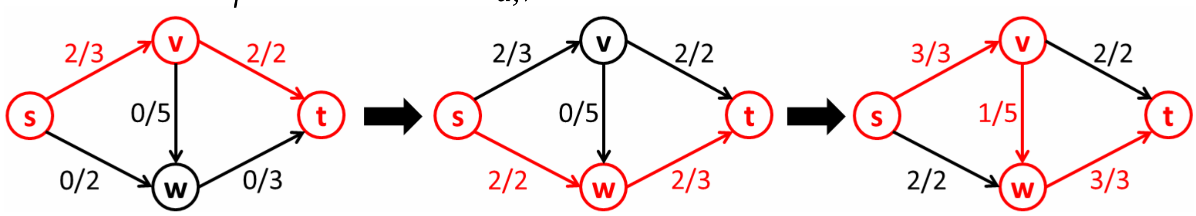
Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: $s,t\in V$, A_G adjacency/weight matrix $A_G \in \boxed{\mathbb{N}}^{V\times V}$

Output: $f:E\to\boxed{\mathbb{N}}$ max. flow from s to t

Goal: maximize $|f|:=\sum_{v:(s,v)\in E} f(s,v)$

The residual G_f of a graph G with flow f has edges $E_f := \{\ (u,v) : A_{u,v} > f(u,v)\ \vee\ f(v,u) > 0\ \}$



Ford-Fulkerson: Initialize $f\equiv0$. Correctness? Runtime $O(m\cdot|f|)$

While there $\boxed{\text{exists a path } P = (s=u_1,\ldots u_K=t) \text{ from } s \text{ to } t \text{ in } G_f}$.

Let $\alpha := \min\{\ A_{u_k,u_{k+1}} - f(u_k,u_{k+1}) : k=1\ldots K-1\ \}$ and $f := f+\alpha\cdot P$.

4. Graph Problems *Edmonds-Karp*

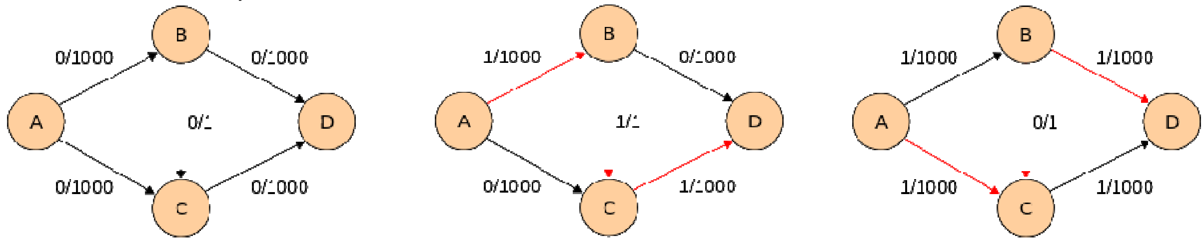
Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: $s,t\in V$, A_G adjacency/weight matrix $A_G \in \boxed{\mathbb{N}}^{V\times V}$

Output: $f:E\to\boxed{\mathbb{N}}$ max. flow from s to t

Goal: maximize $|f|:=\sum_{v:(s,v)\in E} f(s,v)$

The residual G_f of a graph G with flow f has edges $E_f := \{\ (u,v) : A_{u,v} > f(u,v)\ \vee\ f(v,u) > 0\ \}$



Edmonds-Karp: Initialize $f\equiv0$. Runtime $O(n\cdot m^2)$

While there exists a shortest path $P = (s=u_1,\ldots u_K=t)$ from s to t in G_f

Let $\alpha := \min\{\ A_{u_k,u_{k+1}} - f(u_k,u_{k+1}) : k=1\ldots K-1\ \}$ and $f := f+\alpha\cdot P$.

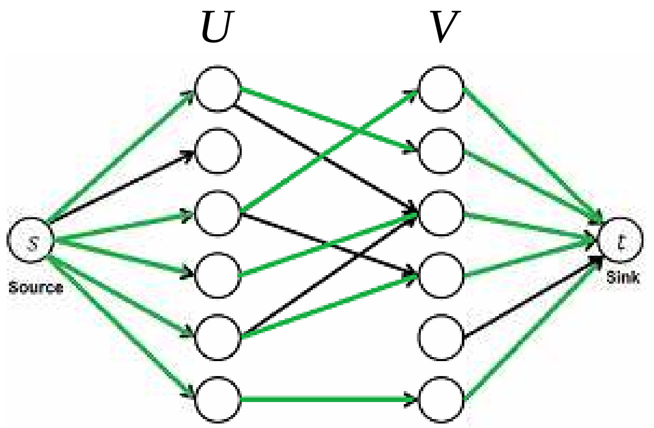
4. Graph Problems

max. Bipartite Matching

Specification: Bipartite graph $G=(U,V,E)$

Input: A_G adjacency/weight matrix $A_G \in \mathbb{N}^{\boxed{U\times V}}$

Output: $F\subseteq E$ max. (weighted) matching



Edmonds-Karp: Initialize $f\equiv 0$. shortest

While there exists a path P from s to t in G_f

Let $\alpha := \min\{\, A_{u_k,u_{k+1}} - f(u_k,u_{k+1}) : k=1\ldots K-1 \,\}$ and $f := f+\alpha\cdot P$.

Runtime $O(n\cdot m^2)$

4. Graph Problems

Min Cut

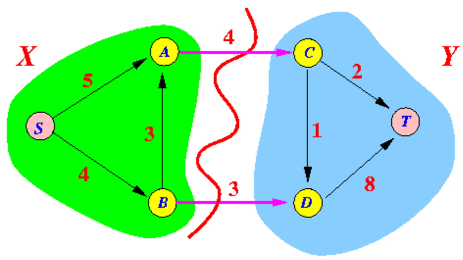
Specification: Graph $G=(V,E)$, $n=\#V$ vertices, $m=\#E$ edges

Input: $s,t\in V$, A_G adjacency/weight matrix $A_G \in \mathbb{N}^{V\times V}$

Output: $C\subseteq E$ min.cut between s,t

Def: A cut from s to t in G is a subset $C\subseteq V$ s.t. $s\in C$, $t\notin C$.

It has capacity $\lambda(C) = \sum_{\substack{(u,v)\in E \\ u\in C, v\notin C}} A_{u,v}$

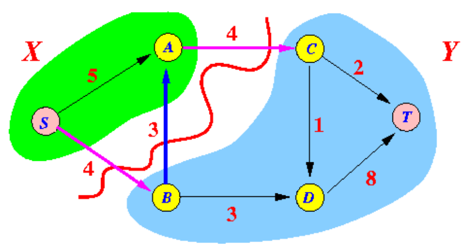


capacity of cut = 4 + 3 = 7

Theorem: $\min_{C \text{ cut } (s,t)} \lambda(C) = \max_{f \text{ flow } (s,t)} |f|$

Proof "≥": For every C,f: $\lambda(C) \geq |f|$.

"≤": Consider $C\subseteq V$ all vertices reachable from s in G_f for max. f from Ford-Fulkerson.



capacity of cut = 4 + 4 = 8