„Introduction to Algorithms"

Syllabus

5. String Problems

- Recap on Strings
- Pattern Matching: Knuth-Morris-Pratt
- Longest Common Substring
- Edit Distance
- Context-free Parsing: Cocke-Younger-Kasami
- Huffman Compression

Martin Ziegler

5. String Problems

strings recap

Specification: Fix finite alphabet $\Sigma \neq \emptyset$, often {**0**,**1**}

A string over Σ is a finite sequence $s = (s_0, \ldots s_{n-1}) \in \Sigma^*$,
input/output as array $s[0 \ldots n\text{-}1]$.

Terminology: Length $|(s_0, \ldots s_{n-1})| = n$,

concatenation $s \circ t$

initial segment $(s_0, \ldots s_{n-1})_{<m} = (s_0, \ldots s_{m-1})$ for $m \leq n$.

s substring of t $\Leftrightarrow$ $\exists u,v$: $t = v \circ s \circ u$

Specification (cont.): Fix finite set $V \neq \emptyset$ disjoint to Σ.

5. String Problems — Pattern Matching

Input: Two strings w and p of lengths $n = |w| >> |p| = m$.

Output: Does w contain p, and where (first, all) ?

arrays $w[0\ldots n\text{-}1]$ and $p[0\ldots m\text{-}1]$

w = ABCXABCDABXABCDABCDABDE

p = ABCDABD

Naïve algorithm:

For k:=0 to n-1
  If $w[k]=p[0]$ then
    Compare $w[k+1\ldots k+m\text{-}1]$ to $p[1\ldots m\text{-}1]$
    If agree, then output k.

runtime $O(n\cdot m)$

Preprocess pattern:

$T[]$ =
-1 0 0 0 -1 0 2 0
A B C D A B D

5. String Problems — Knuth-Morris-Pratt

Input: Two strings w and p of lengths $n = |w| >> |p| = m$.

Output: Does w contain p, and where (first, all) ?

arrays $w[0\ldots n\text{-}1]$ and $p[0\ldots m\text{-}1]$

w = ABCXABCDABXABCDABCDABDE

p = **ABACABABA**

KMP algorithm:

k:=0; j:=0; While k<n do
  If $w[k]=p[j]$ then
    k++; j++;
    If $j=m$ then output k-j; j:=$T[j]$; endif
  else j:=$T[j]$; If j<0 then k++; j++; endif

runtime $O(n+m)$

Preprocess pattern:

$T[]$ =
-1 0 -1 1 -1 0 -1 3 -1 3
A B A C A B A B A

runtime $O(m)$

5. String Problems

Longest Common Substring

Specification: Fix alphabet Σ **Input:** $v\in\Sigma^n$, $w\in\Sigma^m$

Output: Length/positions of longest common substring?

Example: "ABABC" and "BABCA" share "BABC" as longest common substring

Naïve Algorithm:

Try all possible pairs of initial positions

i=0,…n-1 and j=0,…m-1.

For each compare $v[i,..i+k]$ to $w[j,..]$

runtime $O(n{\cdot}m{\cdot}\min(n,m))$

5. String Problems

Longest Common Substring

Specification: Fix alphabet Σ **Input:** $v\in\Sigma^n$, $w\in\Sigma^m$

Output: Length/positions of longest common substring?

Better Algorithm:

Fill integer table LCS[0…n,0…m],
such that LCS[i,j] := length of
longest common suffix
shared by initial segments
$v[0...i-1]$ and $w[0...j-1]$

runtime $O(n{\cdot}m)$

Example:

	A	B	A	B
B	0	1	0	1
A	1	0	2	0
B	0	2	0	3
A	1	0	3	0

$$\text{LCS}[0,j] = 0 = \text{LCS}[i,0]$$

$$\text{LCS}[i+1,j+1] = \text{LCS}[i,j]+1 \quad \text{if } v[i]=w[j]$$
$$= 0 \quad \text{if } v[i]\neq w[j]$$

5. String Problems — Edit Distance

Specification: Fix alphabet Σ **Input:** $v\in\Sigma^n$, $w\in\Sigma^m$

Output: Min. # symbol insert/delete op.s converting v into w.

Proposition: This constitutes a metric on Σ^*. runtime $O(n\cdot m)$

Example: "`kitten`" and "`sitting`" have edit distance 5:

`itten`, `sitten`, `sittn`, `sittin`, `sitting`

Wagner-Fischer Algorithm: Fill table $d[0\ldots n,0\ldots m]$
such that $d[i,j]$:= edit distance of $v[0..i\text{-}1]$ and $w[0..j\text{-}1]$

$d[0,j]=j$ $\quad d[i+1,j+1]=d[i,j]$ if $v[i]=w[j]$

$d[i,0]=i$ $\quad = \min\{\ d[i,j+1]+1\ ,\ d[i+1,j]+1\ \}$ if $v[i]\neq w[j]$

Variants: Dis/allow (i) replacement, (ii) transposition, (iii) …
Assign positive weights to different operations.

5. String Problems — Grammar

Specification: Fix alphabet Σ, disjoint finite set V of variables and fix a finite set R of rules as well as $S\in V$

Input: $w\in\Sigma^*$. **Output:** Can w be generated from S ?

Example: $V=\{S,X\}$, $\Sigma=\{\mathbf{a},\mathbf{b},\mathbf{c}\}$

three rules $S\to\mathbf{a}X S\mathbf{c}$, $S\to\mathbf{abc}$, $X\mathbf{a}\to\mathbf{a}X$, $X\mathbf{b}\to\mathbf{bb}$

generate precisely the strings $\mathbf{a}^n\mathbf{b}^n\mathbf{c}^n$, $n\in\mathbb{N}$.

Definition: A rule r is an assignment $x\to y$,
where $x,y\in(\Sigma\cup V)^*$ and x contains some variable.

A rule $x\to y$ is context-free, if $x\in V$.

5. String Problems — Cocke-Younger-Kasami

Specification: Fix alphabet Σ, disjoint finite set V of variables and fix a finite set R of *context-free* rules as well as $S\in V$

Input: $w\in\Sigma^*$. **Output:** Can w be generated from S ?

Rules in *Chomsky normal form*:

either (i) $\mathrm{X}\to\mathrm{YZ}$ (one to two variables)

or (ii) $\mathrm{X}\to\mathbf{a}$ (one variable to one symbol)

or (iii) $\mathrm{S}\to\varepsilon$ (empty string) [exception only to generate ε..]

Definition: A rule r is an assignment $x\to y$, where $x,y\in(\Sigma\cup V)^*$ and x contains some variable.

A rule $x\to y$ is context-free, if $x\in V$.

5. String Problems — Cocke-Younger-Kasami

Rules of the form (i) $\mathrm{X}\to\mathrm{YZ}$ or (ii) $\mathrm{X}\to\mathbf{a}$ or (iii) $\mathrm{S}\to\varepsilon$

Table $P[s,l,X]$:= "$w_s,\ldots w_{s+l-1}$ can be generated from variable X".

Input: $w\in\Sigma^n$. **Output:** Can w be generated from S ?

runtime $O(n^3)$

```
Initialize P[..] with false.
For each s = 1 to n
  For each rule X → w_s of type (ii)
    P[s,1,X] := true
For l := 2 to n               // Length of span
  For s := 1 to n-l+1         // Start of span
    For k := 1 to l-1         // Partition of span
      For each rule of type (i) X → Y Z
        if P[s,k,Y] and P[s+k,l-k,Z]
          then  P[s,l,X] := true
// w can be generated  iff  P[1,n,S] = true.
```

5. String Problems

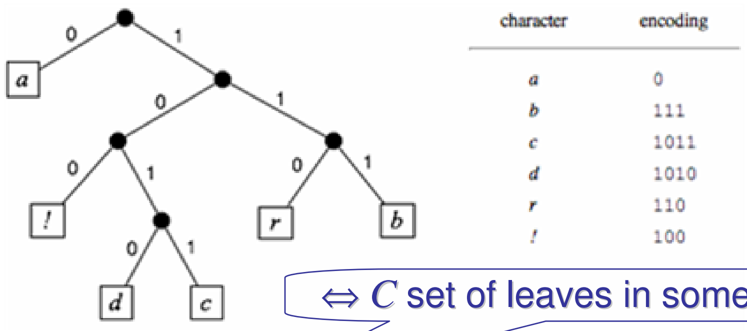
Lossless Compression

Specification: Fix alphabet Σ

Input: $w\in\Sigma^*$ **Output:** "short bit-encoding" of w

1. Determine frequencies f_s of symbols $s\in\Sigma$ in w

"this is an example of a huffman tree"



$\Leftrightarrow C$ set of leaves in some bin.tree

Variable length code, need delimiters—or better:
$C\subseteq\Sigma^*$ is *prefix-free* if $v,w\in C$ and $v \blacktriangleleft w \Rightarrow v=w$.

Char	Freq	Code
space	7	111
a	4	010
e	4	000
f	3	1101
h	2	1010
i	2	1000
m	2	0111
n	2	0010
s	2	1011
t	2	0110
l	1	11001
o	1	00110
p	1	10011
r	1	11000
u	1	00111
x	1	10010

5. String Problems

Prefix Codes

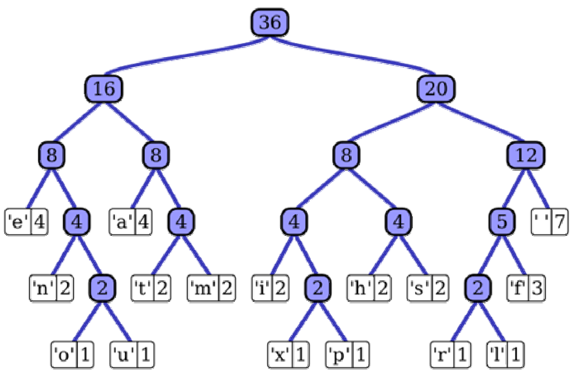
Specification: Fix alphabet Σ

Input: $w\in\Sigma^*$ **Output:** "short bit-encoding" of w

1. Determine frequencies f_s of symbols $s\in\Sigma$ in w

2. Choose prefix-free $C\subseteq\{\mathbf{0},\mathbf{1}\}^*$ / binary tree T

3. Assign to each $s\in\Sigma$ a unique $c_s\in C$ / leaf l_s of T such as to minimize expected length $\sum_{s\in\Sigma} \mathrm{d}(l_s)\cdot f_s$



Char	Freq	Code
space	7	111
a	4	010
e	4	000
f	3	1101
h	2	1010
i	2	1000
m	2	0111
n	2	0010
s	2	1011
t	2	0110
l	1	11001
o	1	00110
p	1	10011
r	1	11000
u	1	00111
x	1	10010

5. String Problems

Huffman Tree

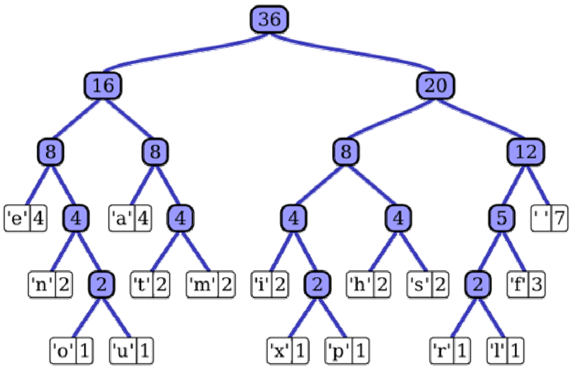
Specification: Fix alphabet Σ

Input: $w\in\Sigma^*$ **Output:** "short bit-encoding" of w

1. Determine frequencies f_s of symbols $s\in\Sigma$ in w

2. Choose prefix-free $C\subseteq\{\mathbf{0,1}\}^*$ / binary tree T

3. Assign to each $s\in\Sigma$ a unique $c_s\in C$ / leaf l_s of T such as to minimize expected length $\sum_{s\in\Sigma} \mathrm{d}(l_s)\cdot f_s$



Idea: Frequent symbols s (=large f_s) should receive small depth $\mathrm{d}(l_s)$, rare ones can afford large depth.

5. String Problems

Huffman Tree

Specification: Fix alphabet Σ

Input: $w\in\Sigma^*$ **Output:** "short bit-encoding" of w

File :

b	p	`	m	j	o	d	a	i	r	u	l	s	e	
1	1	2	2	3	3	3	4	4	5	5	6	6	8	12

Extract two symbols $s,t\in\Sigma$ with least frequencies f_s,f_t.
Combine them to a tree with leaves s,t and root $st:\in\Sigma$ of frequency $f_{st}:=f_s+f_t$.
Repeat.

Idea: Frequent symbols s (=large f_s) should receive small depth $\mathrm{d}(l_s)$, rare ones can afford large depth.