

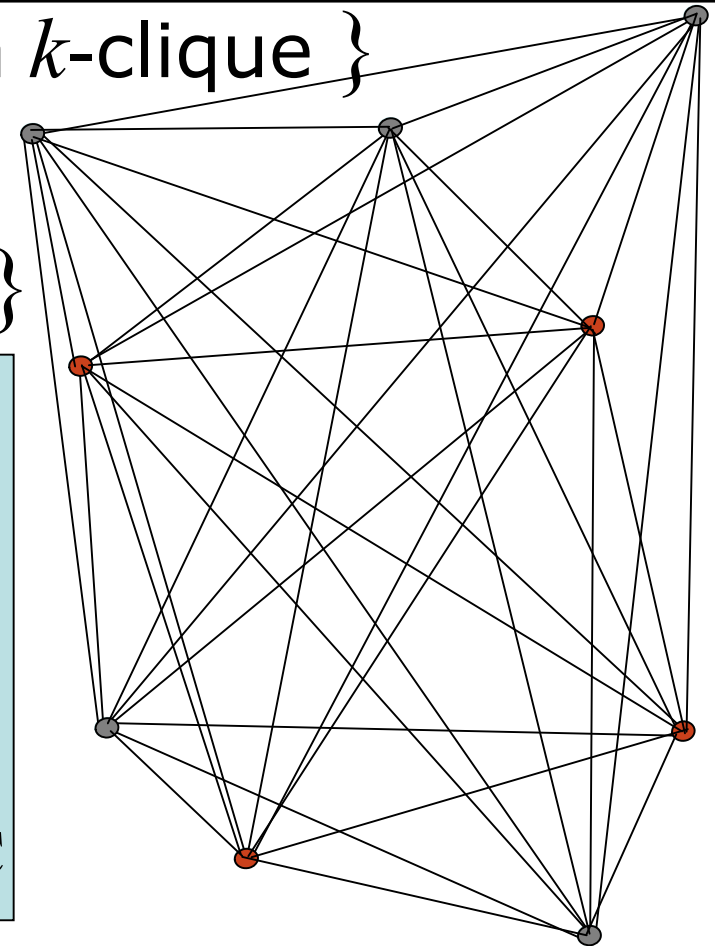
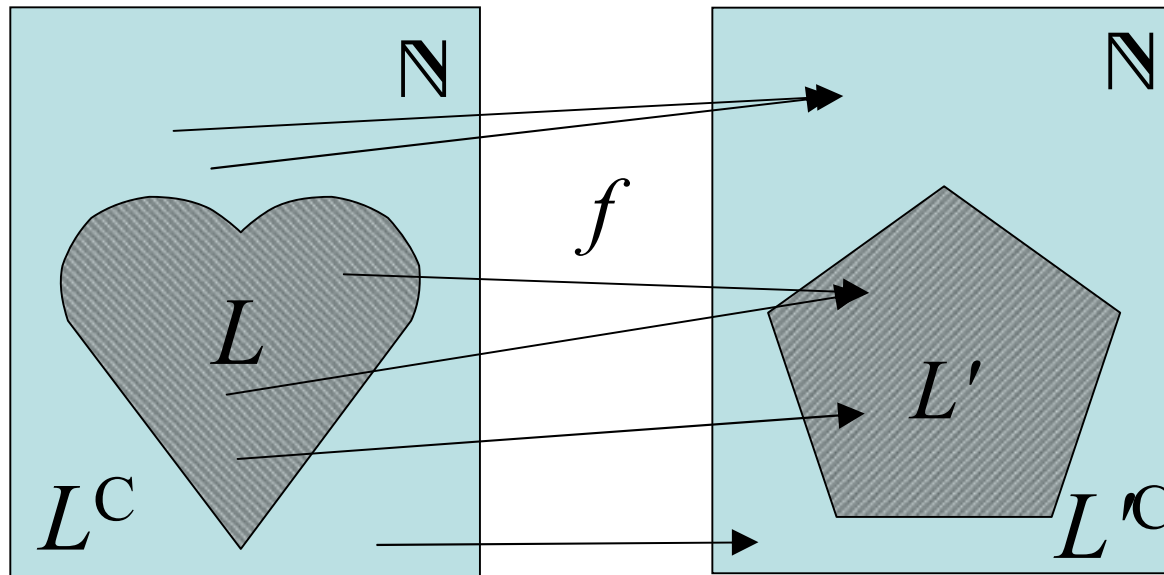
§4 Structural Complexity: \mathcal{NP}_C

- Comparing Problems, quantitatively:
polynomial-time reduction
- Applying reductions:
 $\text{CLIQUE} \equiv_p \text{IS} \preceq_p \text{SAT} \equiv_p \text{3SAT} \preceq_p \text{IS}$
- \mathcal{NP} -completeness
- Master Reduction
- SubsetSum is \mathcal{NP} -complete

Comparing Problems, again

$\text{CLIQUE} = \{ \langle G, k \rangle \mid G \text{ contains a } k\text{-clique} \}$

$\equiv_p \text{IS} = \{ \langle G, k \rangle : G \text{ has } k \text{ pairwise non-connected vertices} \}$



For $L, L' \subseteq \mathbb{N}$ write $L \preceq_p L'$ if exists a polynomial-time computable $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall \underline{x}: \underline{x} \in L \Leftrightarrow f(\underline{x}) \in L'$.

a) $L' \in \mathcal{P} \Rightarrow L \in \mathcal{P} \Rightarrow$ so L . b) $L \preceq_p L' \preceq_p L'' \Rightarrow L \preceq_p L''$

Reduction $IS \leq_p SAT$

Goal: Upon input of (the encoding of) a graph G and $k \in \mathbb{N}$, produce in polynomial time a CNF formula Φ such that:

Φ satisfiable iff G contains $\geq k$ independent vertices

Let G consist of vertices $V = \{1, \dots, n\}$ and edges E .

- Consider Boolean variables $x_{v,i}$, $v \in V$, $i = 1 \dots k$

Vertex v is $\#i$ among the k independent.

There is an i -th vertex

- and clauses $K_i := \bigvee_{v \in V} x_{v,i}$, $i = 1 \dots k$

Vertex v cannot be both $\#i$ and $\#j$.

- and $\neg x_{v,i} \vee \neg x_{v,j}$, $v \in V$, $1 \leq i < j \leq k$

No adjacent vertices are independent.

- and $\neg x_{u,i} \vee \neg x_{v,j}$, $\{u, v\} \in E$, $1 \leq i < j \leq k$

- Length of Φ : $O(k \cdot n + n \cdot k^2 + n^2 k^2) = O(n^2 k^2)$

since $k \leq n$

- Computational cost of $(G, k) \rightarrow \Phi$: polyn. in $n + \log k$

Example Reduction: 4SAT vs. 3SAT

4-SAT: Is formula $\Phi(\underline{Y})$ in 4-CNF satisfiable?

3-SAT: Is formula $\Phi(\underline{Y})$ in 3-CNF satisfiable?

Given $\Phi = (a \vee b \vee c \vee d) \wedge (p \vee q \vee r \vee s) \wedge \dots$

with **literals** $a, b, c, d, p, q, r, s, \dots$

variables,
possibly negated

Introduce new variables u, v, \dots and consider

$\Phi' := (a \vee b \vee u) \wedge (\neg u \vee c \vee d)$
 $\wedge (p \vee q \vee v) \wedge (\neg v \vee r \vee s) \wedge \dots$

$f: \langle \Phi \rangle \rightarrow \langle \Phi' \rangle$

For $L, L' \subseteq \mathbb{N}$ write $L \leq_p L'$ if exists a **polynomial-time** computable $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall \underline{x}: \underline{x} \in L \Leftrightarrow f(\underline{x}) \in L'$.

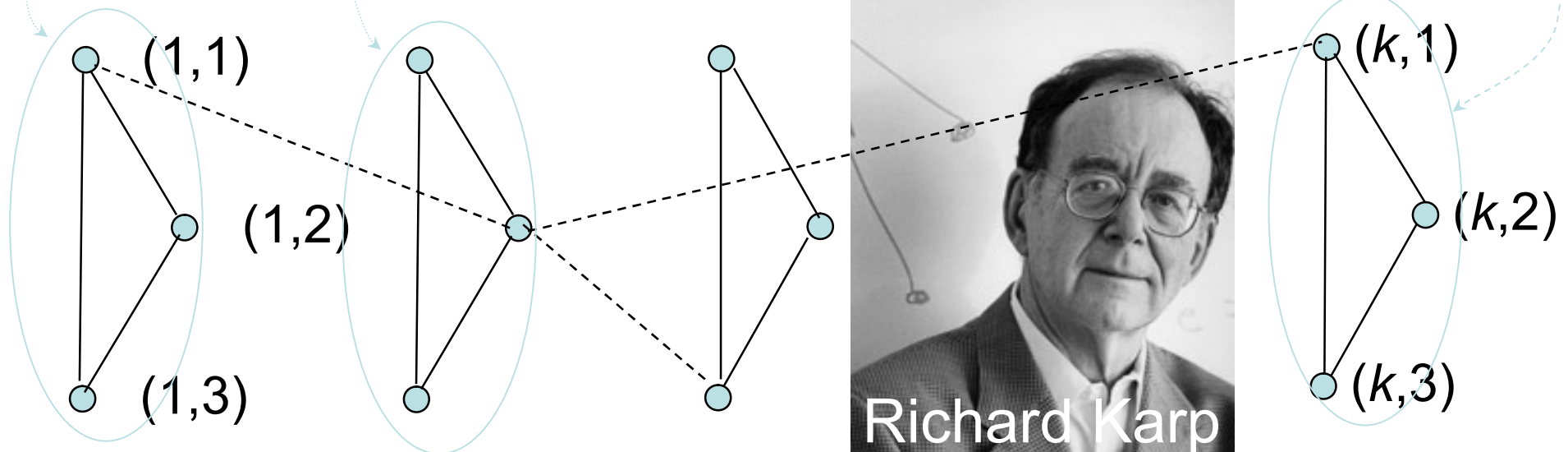
Reduction $3SAT \leq_p IS$

Produce, given a 3-CNF term Φ , within polynomial time a graph G and integer k such that it holds: Φ is satisfiable iff G contains k pairwise non-adjacent vertices.

e.g. $(u \vee \dots \vee \dots) \wedge (\dots \vee \neg u \vee \dots) \wedge (\dots \vee \dots \vee u) \wedge (u \vee \dots \vee \dots)$

$\Phi = C_1 \wedge C_2 \dots \wedge C_k$, $C_i = x_{i1} \vee x_{i2} \vee x_{i3}$, x_{is} literals

$V := \{ (i,1), \dots, (i,3) : i \leq k \}$, $E := \{ \{(i,s), (j,t)\} : i=j \text{ or } \bar{x}_{is} = x_{jt} \}$



Problems of similar complexity

unknown yet

- Showed: $\text{CLIQUE} \equiv_p \text{IS} \preceq_p \text{SAT} \equiv_p \text{3SAT} \preceq_p \text{IS}$.
- These 4 problem have about same complexity:
 - Either all are belong to \mathcal{P} , or none of them.
- Will show: Also TSP, HC, VC and many further problems in \mathcal{NP} belong to this class called \mathcal{NPC} .
- **And** will show: These are 'hardest' problems in \mathcal{NP} .

Cook-Levin Theorem: $\forall L \in \mathcal{NP} : L \preceq_p \text{SAT}$.

Either all or none of the problems in \mathcal{NPC} can be *decided* in polynomial time. In the first case:

A deterministic **WHILE+** program could simulate any *non-deterministic* one with polynomial slowdown!

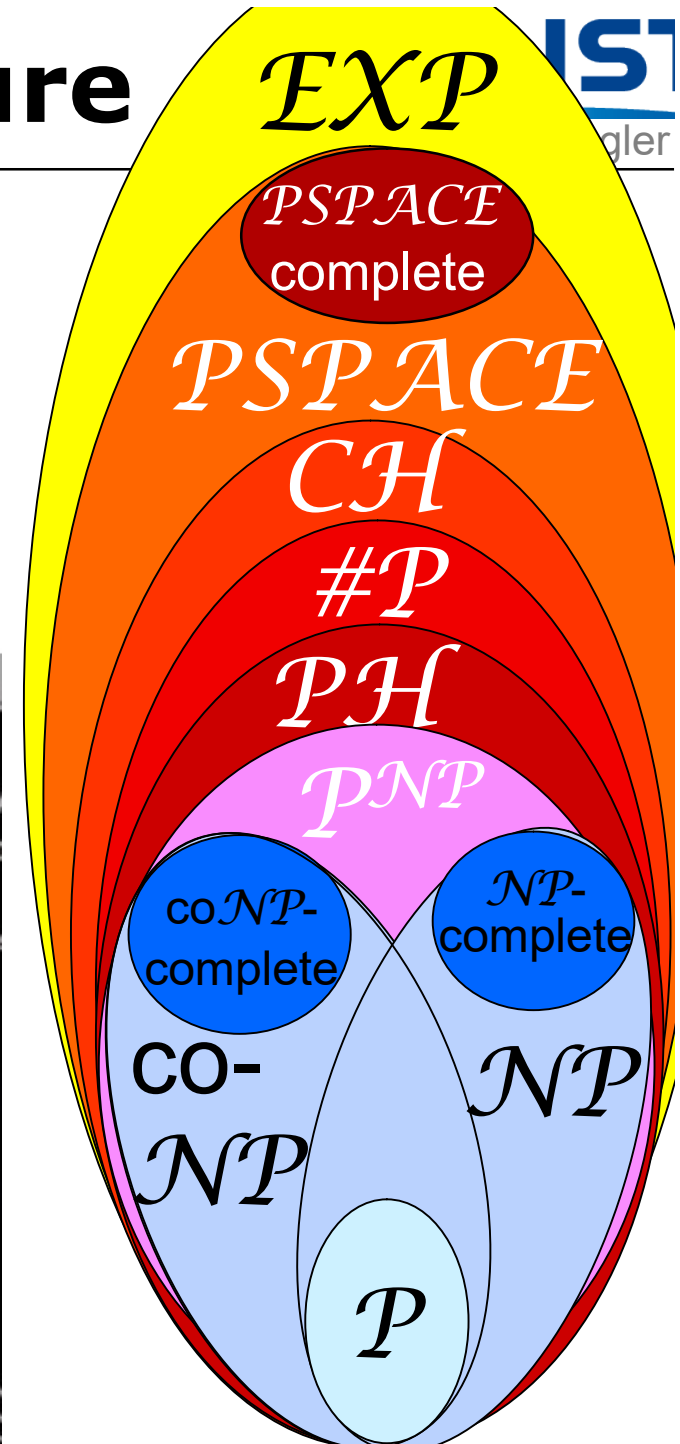
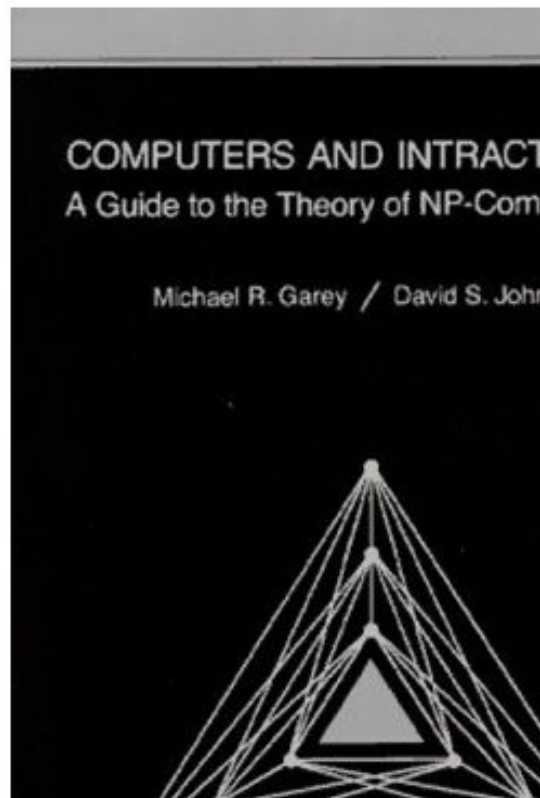
Complexity Class Picture

Def: $A \in \mathcal{NP}$ is \mathcal{NP} -complete if $L \preceq_p A$ holds for every $L \in \mathcal{NP}$.

Theorem (Cook'72/Levin'71):
SAT is \mathcal{NP} -complete!

Lemma: For A \mathcal{NP} -complete and $A \preceq_p B \in \mathcal{NP}$, B is also \mathcal{NP} c.

Now know ≈ 500 natural problems \mathcal{NP} -complete...



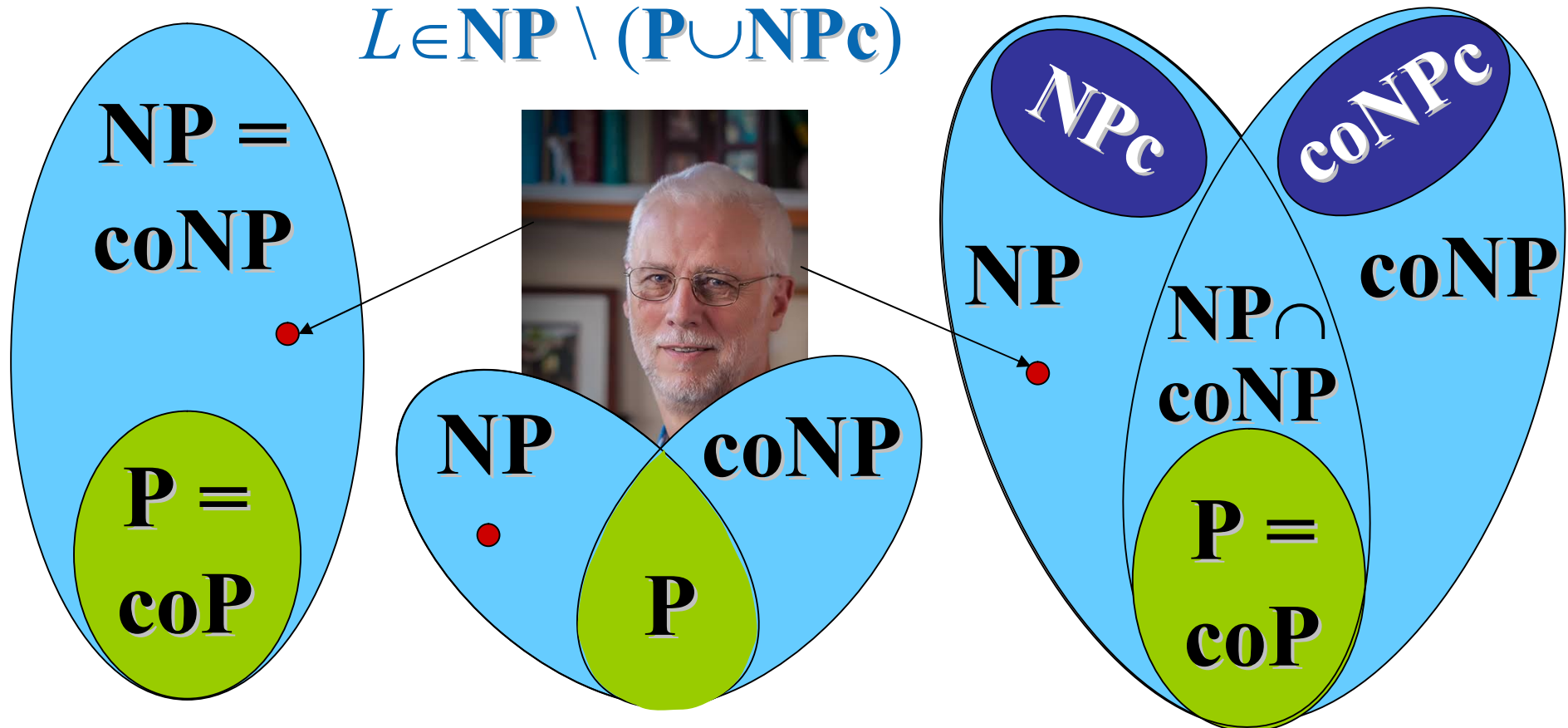
Scenarios for $\mathcal{P} \neq \mathcal{NP}$

$$\text{coNP} := \{ L \subseteq \mathbb{N} : \mathbb{N} \setminus L \in \text{NP} \}$$

$$\text{unSAT} := \{ \varphi \text{ Boolean term, } \forall \underline{x}: \varphi(\underline{x})=0 \} \in \text{coNPc}$$

Theorem [Ladner'75]: If $\mathbf{P} \neq \mathbf{NP}$, there exists

$$L \in \text{NP} \setminus (\mathbf{P} \cup \text{NPc})$$



Master Reduction

$$\text{SAT} = \{ \langle \Phi \rangle : \Phi \text{ Boolean term, } \exists y_1, \dots, y_m : \Phi(y_1, \dots, y_m) = 1 \}$$

Cook/Levin Theorem: SAT is \mathcal{NP} -complete!

Proof (Sketch): Fix $L \in \mathcal{NP}$ and $V \in \mathcal{P}$.

Fix **WHILE+** program \mathcal{B} deciding V in time $\text{poly}(n)$.

Express " $\text{bin}(z_0, \dots, z_{k-1}) \in V$ "

$$\mathbb{N} \ni x \rightarrow \Phi := \Psi_{n+k}(\text{bin}(x), \cdot)$$

as Boolean term " $\Psi_k(z_0, \dots, z_{k-1}) = 1$ " of length $\text{poly}(k)$.

Then $\text{bin}(x_0, \dots, x_{n-1}) \in L \Leftrightarrow \exists y_0, \dots, y_{m-1} \in \{0, 1\} : \Psi_{n+k}(\underline{x}, \underline{y}) = 1$.

Thm: The following problem **UNP** is \mathcal{NP} -complete:

$\{ \langle \mathcal{A}, x, 2^M \rangle : \text{nondetermin. WHILE+ program } \mathcal{A} \text{ accepts input } x \text{ within at most } N \text{ steps} \}$

$$L = \{ x \in \mathbb{N} : \exists y, \ell(y) \leq \text{poly}(\ell(x)), \langle x, y \rangle \in V \}, V \in \mathcal{P}$$

SubsetSum is \mathcal{NP} -complete

$$\{ \langle a_1, \dots, a_N, b \rangle \mid a_1, \dots, a_N, b \in \mathbb{N}, \exists \alpha_1, \dots, \alpha_N \in \{0, 1\} : b = \sum_i a_i \cdot \alpha_i \}$$

- $\text{SubsetSum} \in \mathcal{NP}$ ✓ Show: $3\text{SAT} \leq_p \text{SubsetSum}$
- In polyn.time: $3\text{CNF } \Phi \rightarrow A \subseteq \mathbb{N}$ and $b \in \mathbb{N}$ s.t.
- \exists satisf. assignm. of $\Phi \Leftrightarrow \exists B \subseteq A: b = \sum_{a \in B} a$

$$\text{Eg. } \Phi = (x_1 \vee \neg x_3 \vee x_5) \wedge (\neg x_1 \vee x_5 \vee x_4) \wedge (\neg x_2 \vee \neg x_2 \vee \neg x_5)$$

$v_1 :=$	100	10000	$v_1' :=$	010	10000	$b :=$	444	11111
$v_2 :=$	000	01000	$v_2' :=$	002	01000	$c_1 :=$	100	00000
$v_3 :=$	000	00100	$v_3' :=$	100	00100	$d_1 :=$	200	00000
$v_4 :=$	010	00010	$v_4' :=$	000	00010	$c_2 :=$	010	00000
$v_5 :=$	110	00001	$v_5' :=$	001	00001	$d_2 :=$	020	00000
						$c_3 :=$	001	00000

m clauses in n var.s $\rightarrow 2n+2m+1$ values à $n+m$ dec.digits

§4 Summary

- Comparing Problems, quantitatively:
polynomial-time reduction
- Applying reductions:
 $\text{CLIQUE} \equiv_p \text{IS} \preceq_p \text{SAT} \equiv_p \text{3SAT} \preceq_p \text{IS}$
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§4 Quiz

Section 4 has introduced
polynomial-time reduction.



Other parts of this course have
considered other *notions* of reduction.



- a) Report three different *notions* of reduction from this course.
- b) „Invent“ a fourth *notion* of reduction and report your definition of it.