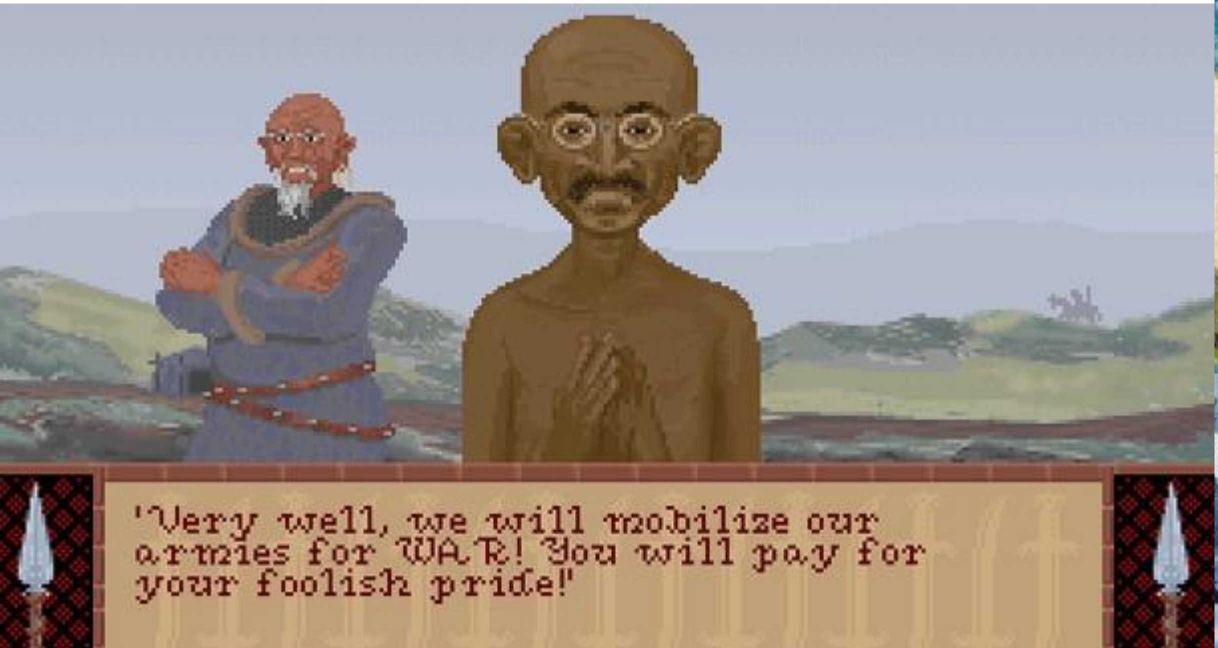
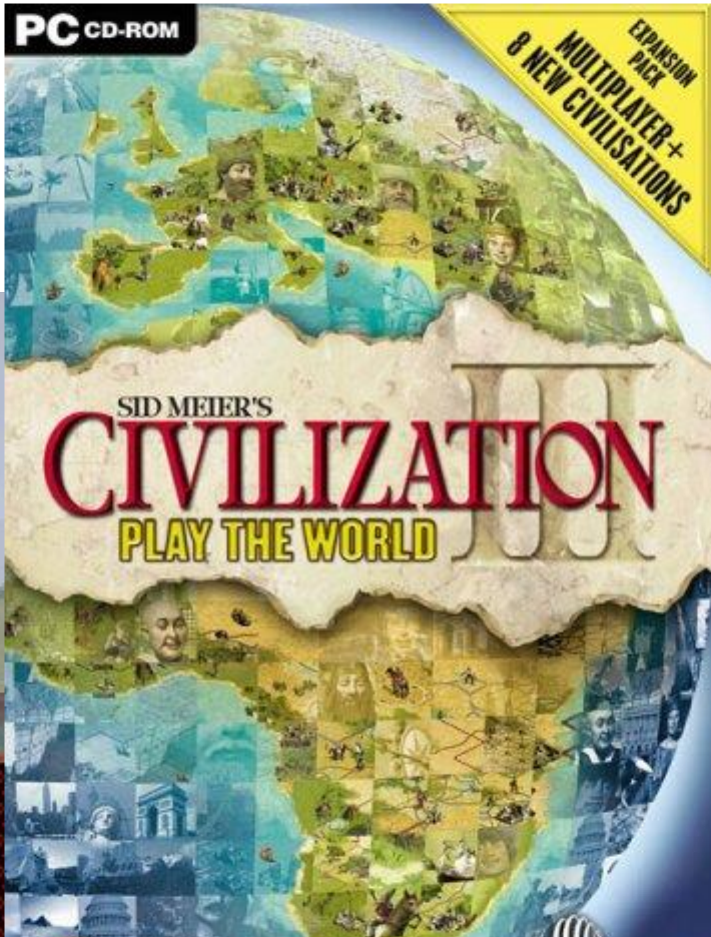
§2 Tree Data Structures

- Abstract Data Types
  - Hide hardware/implementation/data structure
  - Recap: basic / derived/ linked data structures
- AVL Trees:
  - definition, properties
  - operations/maintenance, cost, deficiency
- Binomial Trees, Binomial Heaps:
  - definition, operations, analysis
  - ExtractMin, DecreaseKey, **Merge** in $O(\log n)$

Hardware vs. Math. Data Types

- Each country/leader described by "A.I." character parameters.
- Initially *Gandhi*.**aggression** = 1
- When country adopts democracy, **`aggression -= 2`**.


'Very well, we will mobilize our armies for WAR! You will pay for your foolish pride!'



Abstract Data Types

Integer $\mathbb{Z}$ (vs. byte/word etc.) — $0, 1, +, -, \times,$ div $, >$

Real $\mathbb{R}$ (vs. float/double etc.) — $0, 1, +, -, \times, \div, >$

Stack of X — push x, pop, isEmpty

Queue of X — enque x, deque, isEmpty

(Dynamic) array of X — [], size, (re-size), search

$O(1)$ $O(n)$

$O(\log n)$

Sorted array of Y — search y, insert y, delete y

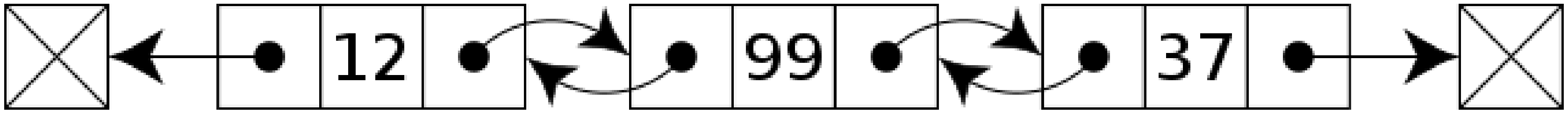
$O(n)$

Priority queue of Y — findmin, insert y

where Y is totally ordered

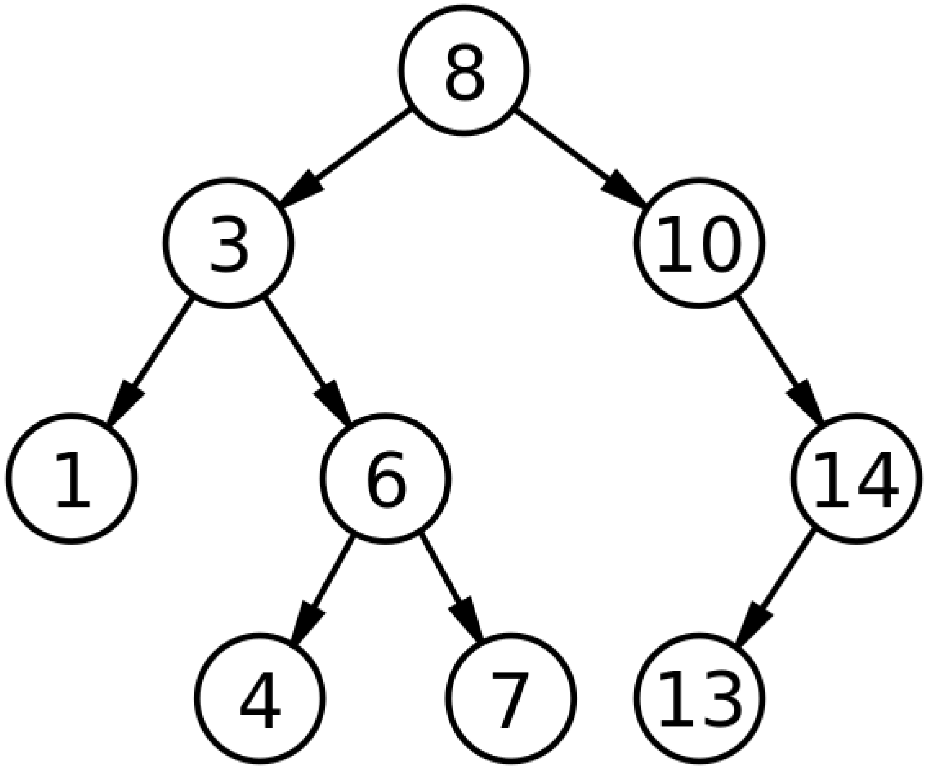
$O(\log n)$

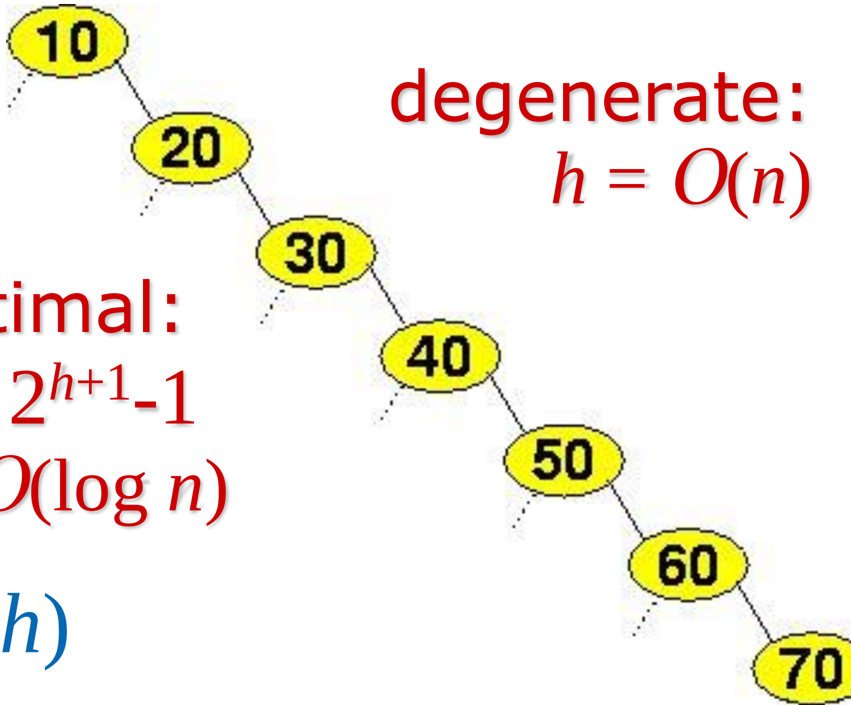
Linked Data Structures



(Doubly) linked list:

- insert, delete in $O(1)$
- search in $O(n)$





degenerate:
$h = O(n)$

optimal:
$n = 2^{h+1}-1$
$h = O(\log n)$

Binary search tree:
search, insert, delete in $O(h)$

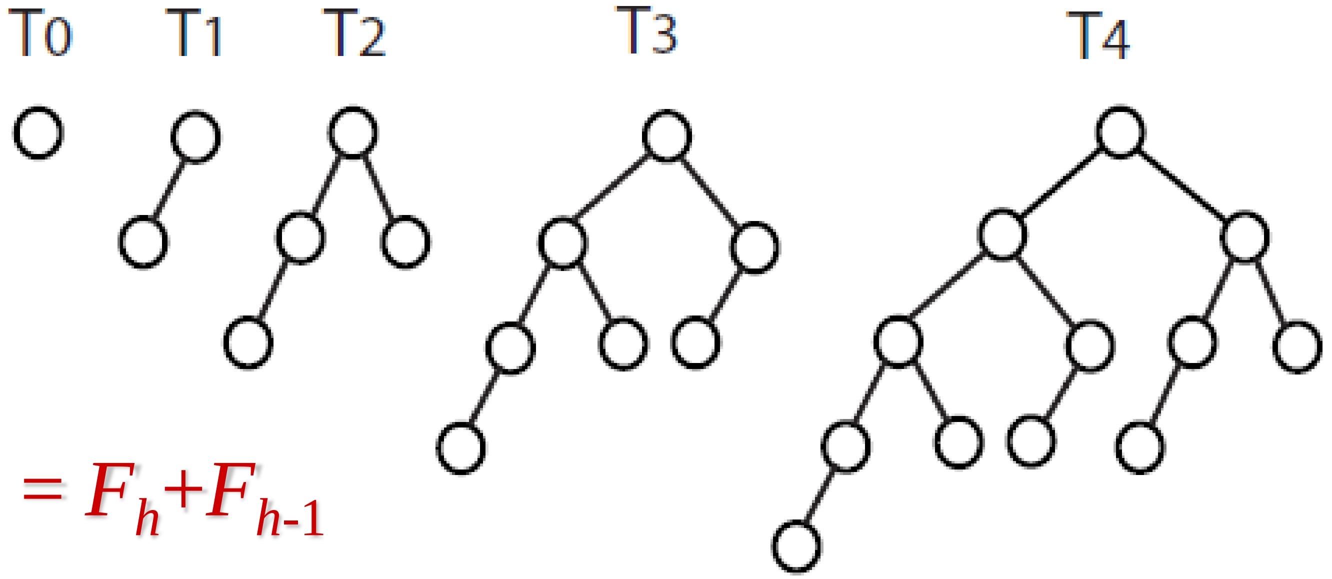
§2 Tree Data Structures

- Abstract Data Types
    - Hide hardware/implementation/data structure
    - Recap: basic / derived/ linked data structures
- AVL Trees:
    - definition, properties
    - operations/maintenance, cost, deficiency
- Binomial Trees, Binomial Heaps:
    - definition, operations, analysis
    - ExtractMin, DecreaseKey, **Merge**

Adelson-Velsky-Landis'62

Binary tree s.t. any two sibling subtrees have height difference at most 1!

minimal AVLTrees:



Fibonacci

$F_0=0,\ \ F_1=1,\ F_{h+1} = F_h+F_{h-1}$

$n(h)$:=min #nodes of AVLTree of height $h \le O(\log n)$

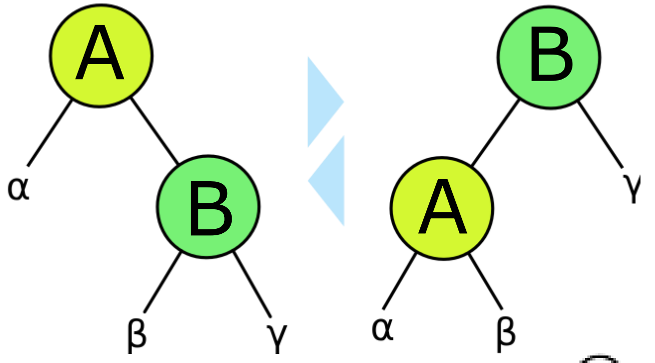
$\#n(0)+1=\boldsymbol{F}_3,\quad \#n(h+1)+1 = \#n(h)+1 + \#n(h\text{-}1)+1 \ \boldsymbol{=F_{h+4}}$

Recall $F_h = (\varphi^h\text{-}(\text{-}1/\varphi)^h)/\sqrt{5} \ \ge \ \Omega(1.6^h) \ \Rightarrow h=O(\log F_h)$

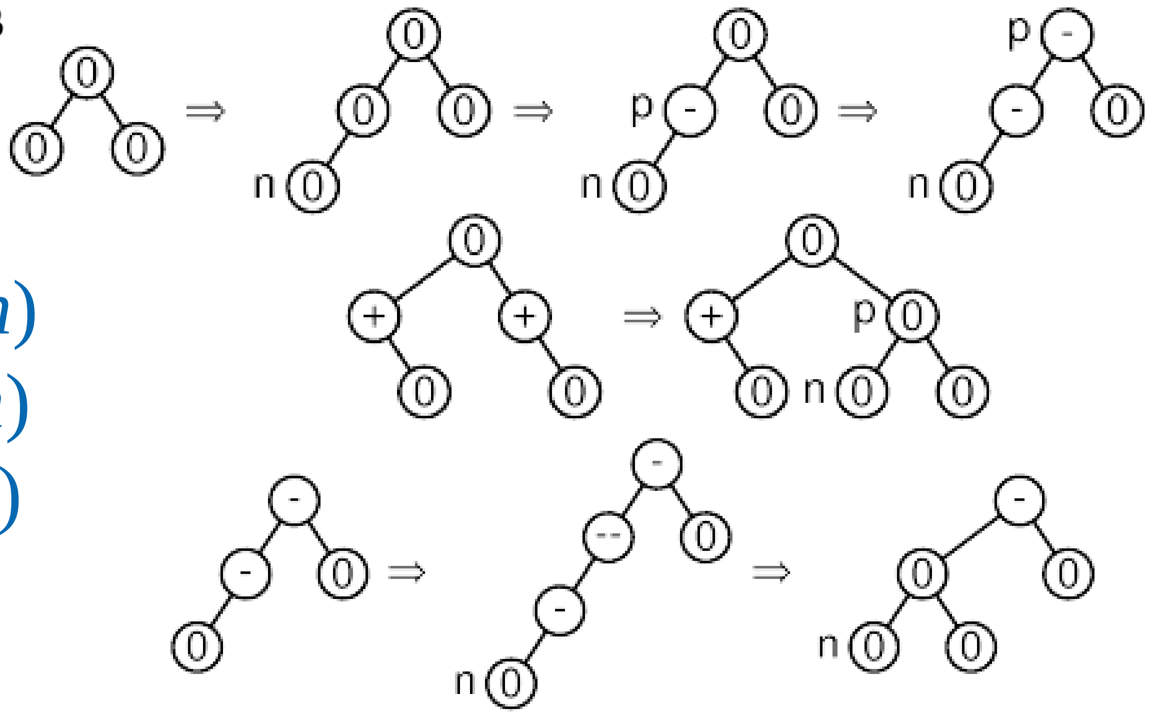
AVL Tree Maintenance

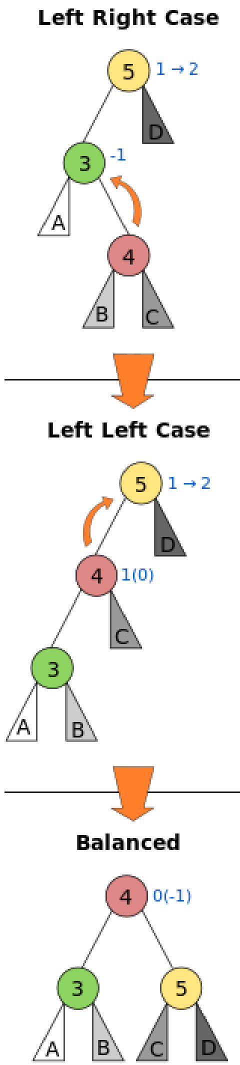
Binary tree s.t. any two sibling subtrees have height difference at most 1!

$h \leq O(\log n)$



Store & recursively update balance indicators **+, 0, –** .
After **insert** at a left leaf, propagate up: three cases

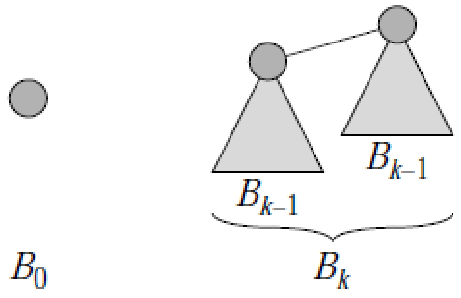




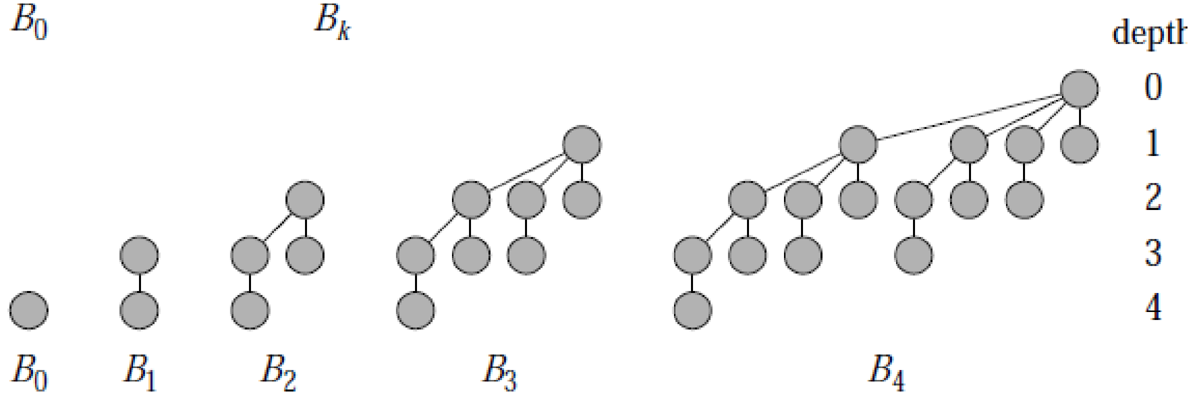
search $O(\log n)$
insert $O(\log n)$
delete $O(\log n)$
merge $O(n)$

Binomial Trees

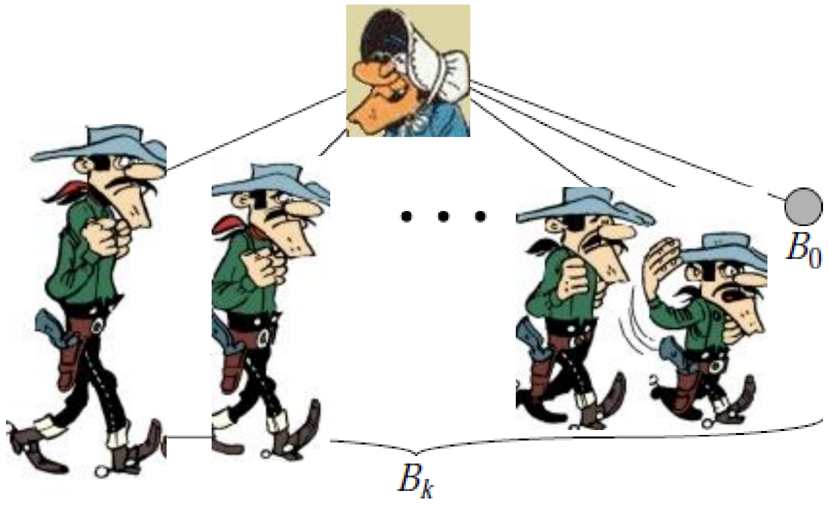
A *binomial tree* is an ordered tree defined recursively:



Merge: $B_k + B_k \to B_{k+1}$



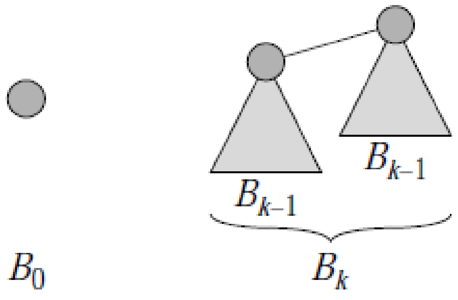
Require and maintain each B to be *heap-ordered*: key(node) ≤ key(children)



Lemma: B_k has $n=2^k$ nodes and height k and maximum degree k. Precisely $\binom{k}{d}$ nodes are at depth d.

Quiz

A *binomial tree* is an ordered tree defined recursively:



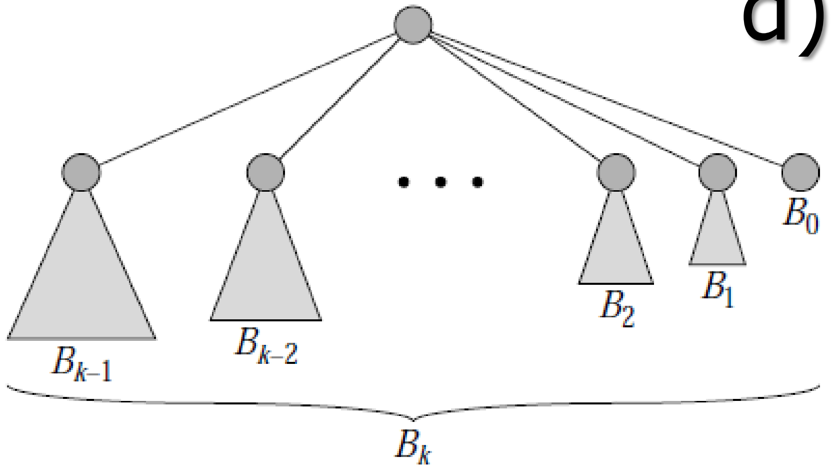
Prove by induction:

a) B_k has $n=2^k$ nodes

b) B_k has height k

c) B_k has maximum degree k

d) B_k = root, connected to children $B_0, B_1 \ldots B_{k-1}$

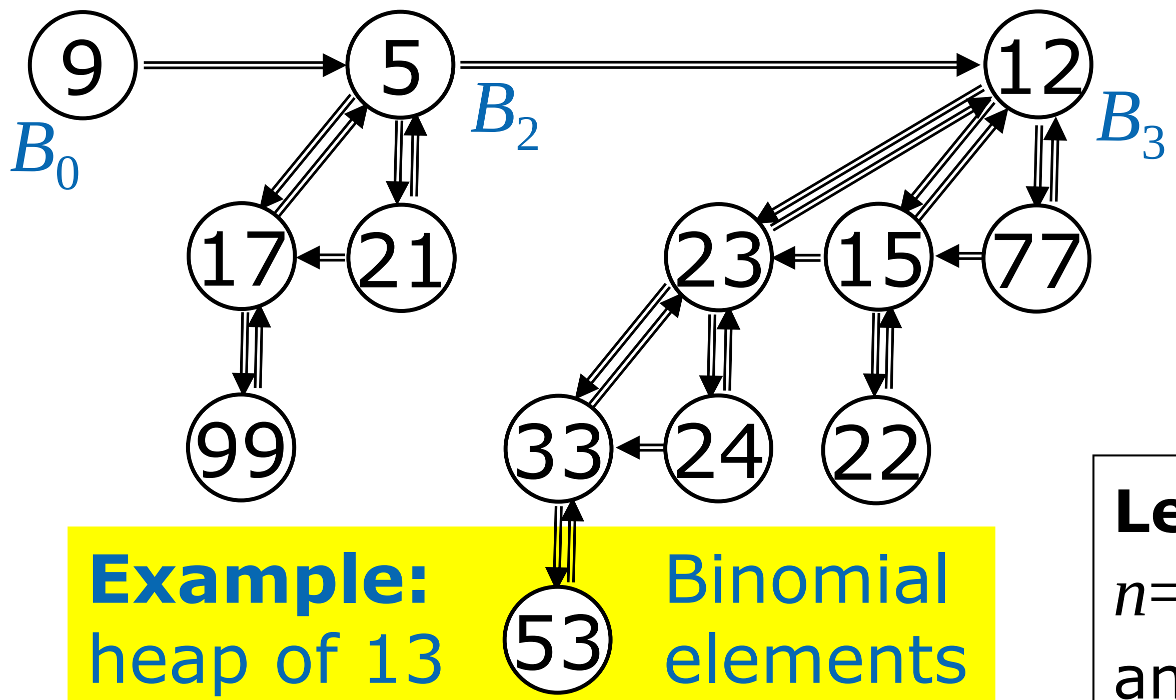


Binomial Heaps

A *binomial tree* is an ordered tree defined recursively.

Binomial heap is ascend. list of binomial trees containing, for each k, at most one B_k.

Require and maintain each B to be *heap-ordered*:
key(node) ≤ key(children)



Example: heap of 13 Binomial elements

Lemma: B_k has $n=2^k$ nodes and height k and maximum degree k.

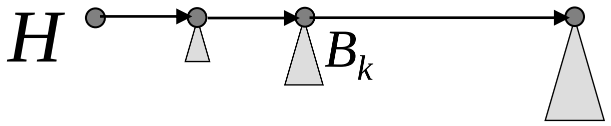
Pointers to: children, parent, left sibbling, next binm. tree

List length, vertex degree, tree depth: all ≤ $O(\log n)$

Operations on Binomial Heaps

A *binomial tree* is an ordered tree defined recursively.

Binomial heap is ascend. list of binomial trees containing, for each k, at most one B_k.



Operations:

1. *Create* one-elem. bin.heap: $O(1)$
2. *Extractmin*(imum): $O(\log n)$
3. *Merge* two binom. heaps: $O(\log n)$
4. *Insert* element: $O(\log n)$
5. *DecreaseKey*: $O(\log n)$
6. *Delete*: $O(\log n)$

Require and maintain each B to be *heap-ordered*:
key(node) ≤ key(children)

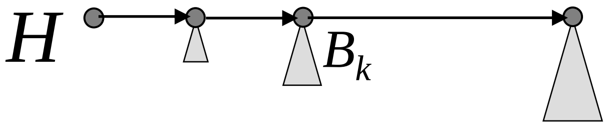
no *search* operation: entries via reference/link/"handle"

List length, vertex degree, tree depth: all ≤ $O(\log n)$

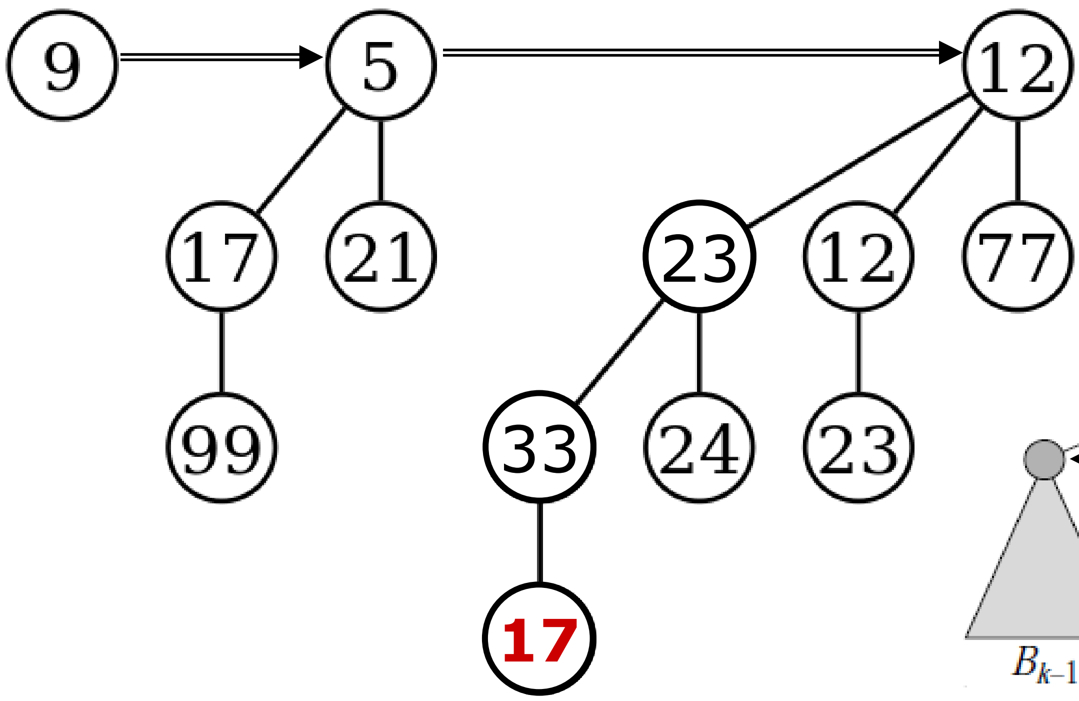
ExtractMin and DecreaseKey

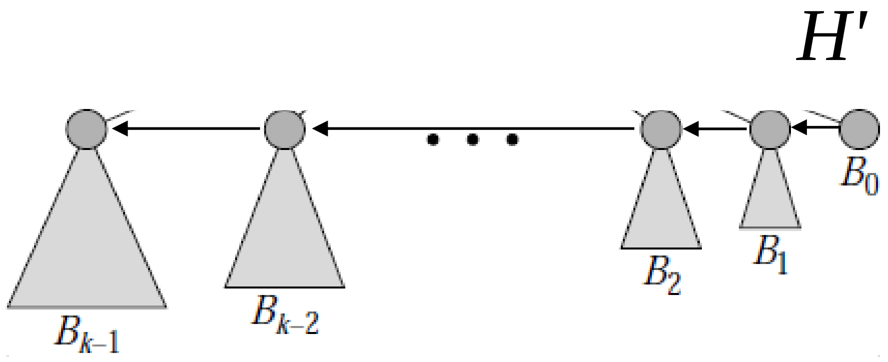
A *binomial tree* is an ordered tree defined recursively.

Binomial heap is ascend. list of binomial trees containing, for each k, at most one B_k.



Require and maintain each B to be *heap-ordered*: key(node) ≤ key(children)

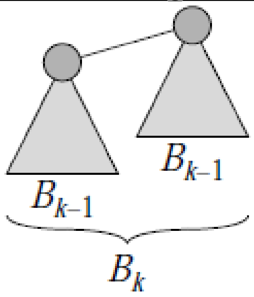




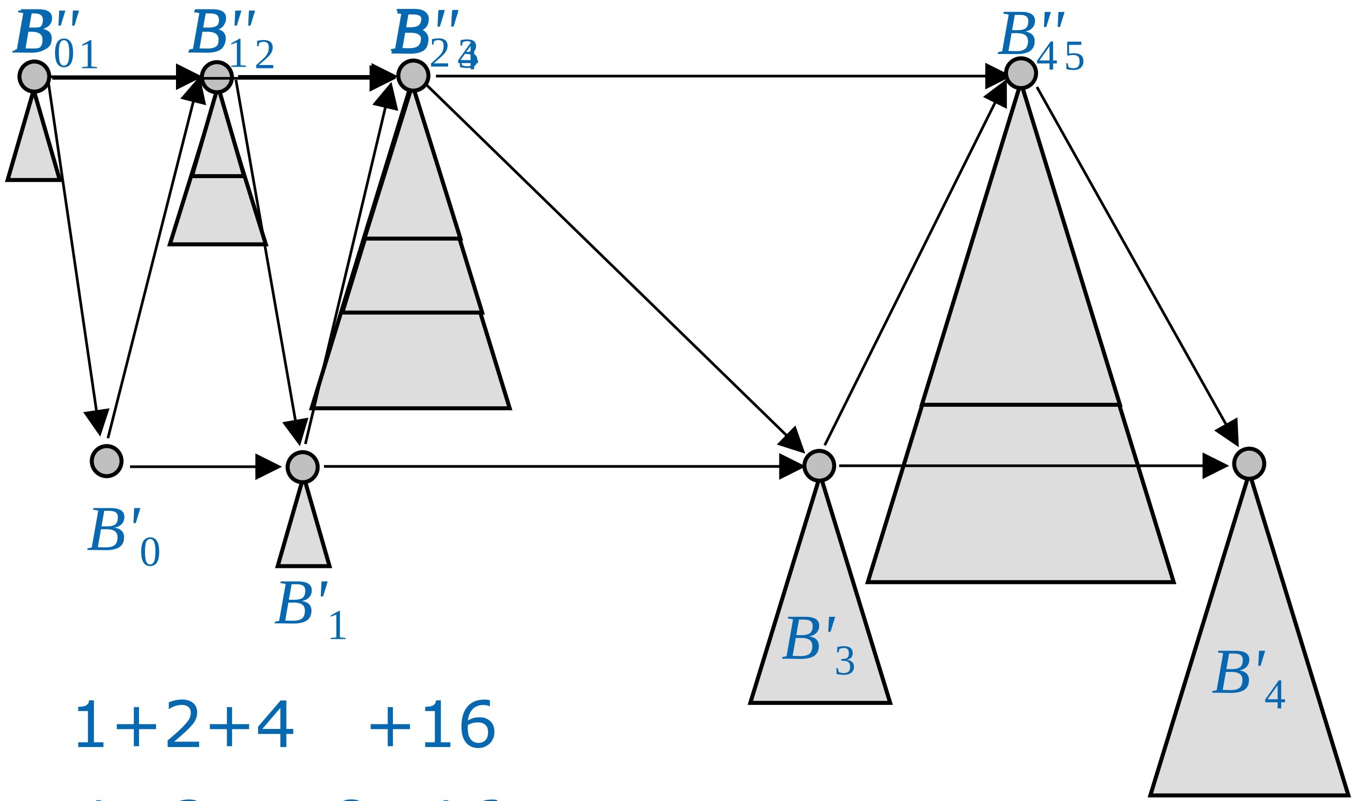
List length, vertex degree, tree depth: all ≤ $O(\log n)$

Merging two Binomial Heaps

- Merging two Binomial *Trees* B_k: cost O(1)



Binomial heap is ascend. list of binomial trees containing, for each k, at most one B_k.



Require and maintain each B to be *heap-ordered*: key(node) ≤ key(children)

Binary addition of integers $\leq n$:
#carries $\leq O(\log n)$

1+2+4 +16
+ 1+2 +8+16
= 2 +16+32 √

List length, vertex degree, tree depth: all ≤ $O(\log n)$

§2 Recap

- Abstract Data Types
  - Hide hardware/implementation/data structure
  - Recap: basic / derived/ linked data structures
- AVL Trees:
  - definition, properties,
  - operations/maintenance, deficiency
- Binomial Trees, Binomial Heaps:
  - definition, operations, analysis
  - ExtractMin, DecreaseKey, **Merge** in $\boxed{O(\log n)}$

worst-case