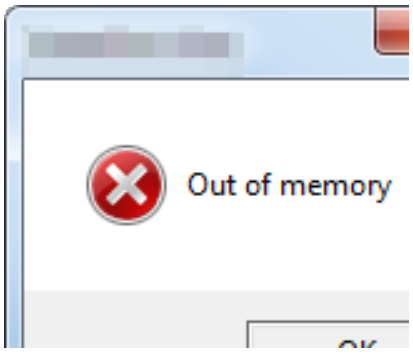
§10 Memory-Efficient Algorithms

- Motivation
- Fibonacci revisited
- Matrix Powering revisited
- Graph Reachability revisited
- Methods/Cost of Saving Memory
- Memory ≈ Parallel Time
- Streaming Algorithms
- (I/O-efficient/*external* memory algorithms)

Motivation

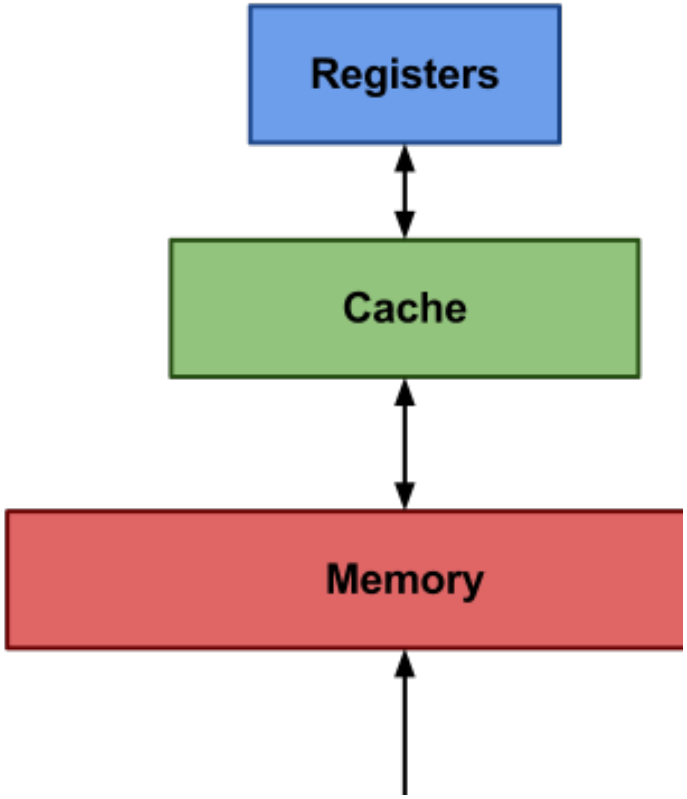
Memory is limited, Time is unlimited.



CPU Registers
100 bytes
< 1ns

Static RAM (SRAM)
megabytes
0.5-2.5ns

Dynamic RAM (DRAM)
gigabytes
50-70ns



- (I/O-efficient/*external* memory algorithms)

Fibonacci, *Revisited*

- Algorithm ***FibRek(n)***:

$T(n) = T(n\text{-}1)+T(n\text{-}2)+O(n) = 2^{O(n)}$ **bit** op.s

$S(n) = \max\{ S(n\text{-}1), S(n\text{-}2) \} + O(n) = O(n^2)$ **bits**

```
if n=0 return 0;
n=1 return 1;
return FibRek(n-1)
      +FibRek(n-2)
```

- Algorithm ***FibIter(n)***:

$T(n) = O(n^2)$ **bit** op.s

$S(n) = O(n)$ **bits**

```
fib := 1; fibL := 0; while n>1
                     fibL := fib;
                     tmp; n := n-1
```

- via 2×2 **integer matrix power**:

$$\begin{vmatrix} F_n \\ F_{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} F_{n-1} \\ F_{n-2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^{K} \cdot \begin{vmatrix} F_{n-K} \\ F_{n-K-1} \end{vmatrix} \qquad K := n-1$$

Reminder: $F_n \approx \varphi^n$ of **bit**-length $O(n)$, $\varphi \approx 1.6$

Integer Matrix Powering

- Algorithm ***FibRek(n)*:**

$T(n) = T(n\text{-}1)+T(n\text{-}2)+O(n) = 2^{O(n)}$ **bit** op.s

$S(n) = \max\{\, S(n\text{-}1), S(n\text{-}2)\,\} + O(n) = O(n^2)$ **bits**

Fast(est) integer multipl. [Harvey& van der Hoeven'19]

- Algorithm ***FibIter(n)*:**

$T(n) = O(n^2)$ **bit** op.s $\qquad S(n) = O(n)$ **bits**

- via 2×2 **integer matrix power** $A \to A^K$, $K := n-1$
- $A \to A^K$ **iteratively:** $T=O(K^2 \cdot \log K)$ $\qquad S=O(K)$ **bits**
- $A \to A^2, A^4, A^8, \ldots$ **repeated squaring:**

$d=2$ $\qquad T=O(K \cdot \log^2 K)$ **bit** op.s, $\qquad S=O(K)$ **bits**

$$\log \max |A^K| \le O(K \cdot \log d + K \cdot \log \max |A|) \qquad \le O(K)$$

Boolean Matrix Power, *entrywise*

Given: Boolean $N \times N$ matrix A

Input: integers $K,I,J < N$

Output: $(A^K)_{I,J}$

does *not* count as *working* memory use

Given: Boolean $N \times N$ matrices X, Y

Input: integers I,J

Output: $(X \cdot Y)_{I,J} = \bigvee_L X_{I,L} \wedge Y_{L,J}$

only $O(\log N)$ **bits** of mem.

Repeated Squaring, recursively:

$A^K = (A^{K/2})^2$ if K even

$A^K = (A^{(K-1)/2})^2 \cdot A$ if K odd

$S(K) = S(\lfloor K/2 \rfloor) + O(\log N)$ **bits**

Use only $O(\log K \cdot \log N)$ **bits** of *working* memory!

Boolean Matrix Power, *explicitly*

```
const bool A[N,N];   // N = dimension

bool powerA(uint K,I,J)  {   // returns  A^K[I,J]
  if (K==0)  return (I==J);      // A^0 = identity
  bool res=false;  uint K2=K/2;  // rounded down
  if (K%2==0)   // K even
    for (uint L=0; L<N; L++)      // O(log N) bits
      res |= powerA(K2,I,L) && powerA(K2,L,J);
  else          // K odd
    for (uint L=0; L<N; L++)      // O(log N) bits
      for (uint M=0; M<N; M++) // O(log N) bits
        res |=  A[I,L]  &&
                powerA(K2,L,M) && powerA(K2,M,J);
return res;  }
```

Graph Reachability *Revisited*

Given directed graph $G=(V,E)$ with $N=|V|$ vertices.

Input vertices $s,t\in V$ and distance $K\in\mathbb{N}$:

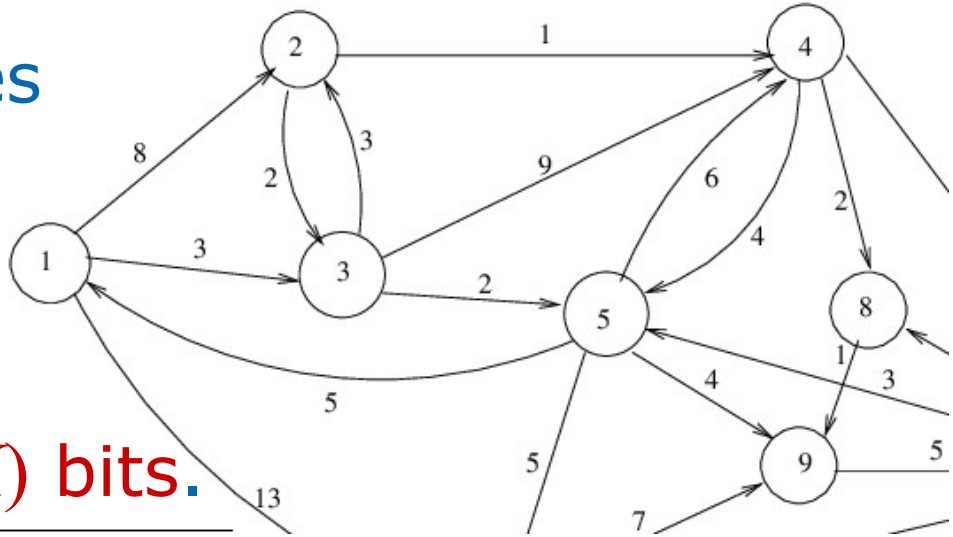
Is t reachable from s in G within at most K steps**?**

Given graph $G=(V,E)$ *not* stored/input,

but accessed via queries "$(u,w)\in E$ **?**"

Dijkstra maintains/updates "*wavefront*" of (tentative) distances $d_s(u)\in\{1...K\}$ to s for each $u\in V$:

Stores N integers à $O(\log K)$ bits.



***Entrywise* $N\times N$ Boolean Matrix Powering** $A\rightarrow A^K$

Use only $O(\log K\cdot\log N)$ **bits** of *working* memory!

Graph Reachability *Summary*

Given directed graph $G=(V,E)$ with $N=|V|$ vertices.

Input vertices $s,t \in V$ and distance $K \in \mathbb{N}$:

Is t reachable from s in G within at most K steps**?**

Graph $G=(V,E)$ *not* stored/input,

but accessed via queries "$(u,w) \in E$ **?**"

Reingold'08: *UN*directed Reachability in $O(\log N)$ bits**!**

***Directed* Graph Reachability** in $O(\log N \cdot \log K)$ bits**!**

Dijkstra stores N integers à $O(\log K)$ bits.

***Entrywise* $N \times N$ Boolean Matrix Powering** $A \to A^K$

Use only $O(\log K \cdot \log N)$ **bits** of *working* memory!

Means/Cost of Memory Saving

"Forget and recalculate intermediate results."

Memory-Saving Approach $A^K = (A^{K/2})^2$, K even

Matrix squaring $(X \cdot X)_{I,J} = \bigvee_L X_{I,L} \wedge X_{L,J}$

now time $T(K) = O(N) \cdot (T(K/2) + T(K/2))$

time $O(N^{\log K} \cdot K)$

***Directed* Graph Reachability** in $O(\log N \cdot \log K)$ bits!

Dijkstra stores N integers à $O(\log K)$ bits

Memory ≈ Parallel Time

Theorem (Borodin'77):

decision

Any algorithm with *seq.* time $T(n)$ using $S(n)$ mem bits can be parallelized to a circuit of *depth* $O(S(n)\cdot\log T(n))$

***Graph Reachability*, distance K among N vertices:**

Recall §9 : *parallel* Depth $= O(\log N \cdot \log K)$

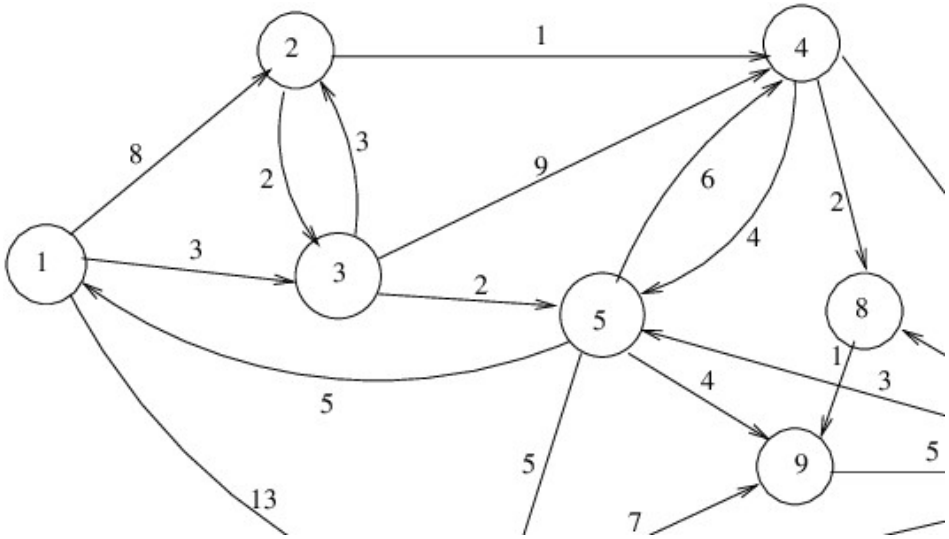
Reachability Method

Theorem (Borodin'77): *decision*

Any algorithm with *seq.* time $T(n)$ using $S(n)$ mem bits can be parallelized to a circuit of *depth* $O(S(n)\cdot\log T(n))$.

Proof Sketch:

Configuration graph
has $N:=2^{O(S(n))}$ vertices.
Unique (sic!) accepting
configuration reached
within $K:=T(n)$ steps. ■

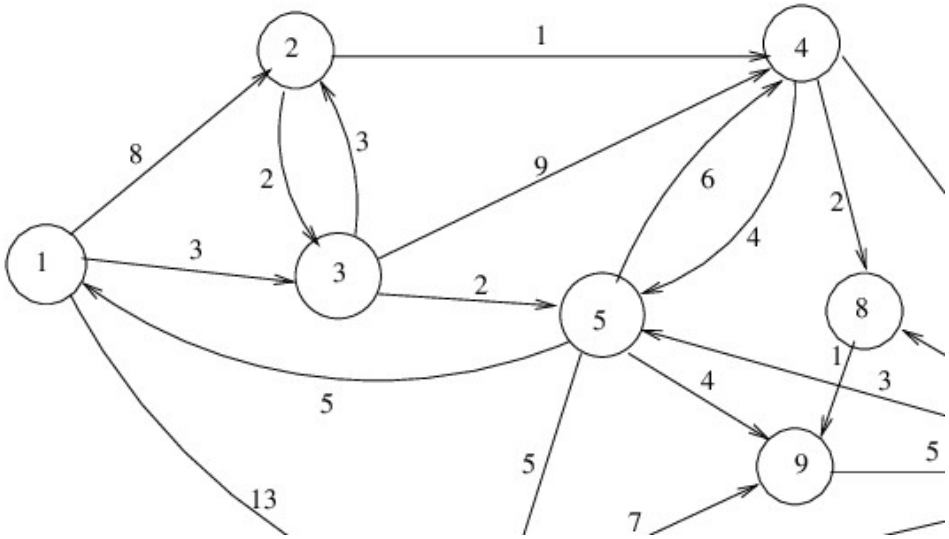


Recall §9 : *parallel* Depth = $O(\log N \cdot \log K)$

Configuration Graph

- *Configuration* of a program/algorithm:
  "snapshot" of its memory/variables + prog.counter
- *Start configuration* contains stored input,
- *End configurations* w.l.o.g. after erasing memory:
  one accepting, one rejecting (decision problem)

Configuration graph
has $N := 2^{O(S(n))}$ vertices.
Unique (sic!) accepting
configuration reached
within $K := T(n)$ steps.



- *Step* of progr./algorithm = *directed* edge in graph

Streaming Problems

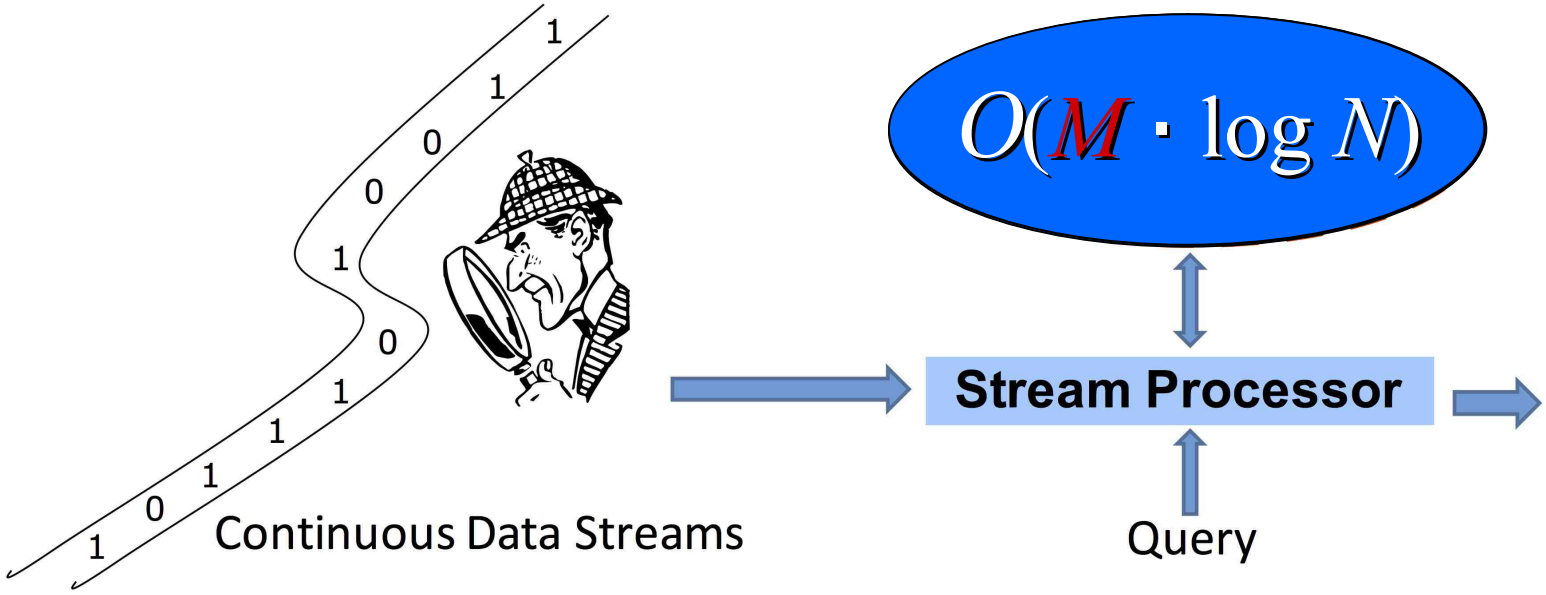
Example: Input sequence $x_n \in \{0,\ldots,M\text{-}1\}$, $n=0..N\text{-}1$

a) Count: Given $y \in \{0,\ldots,M\text{-}1\}$, output $\#\{n: x_n=y\} =: C_y$

b) Sum: output $\sum_n x_n^k$

c) Counting *Distinct*: output $F_0 = \#\{ y: C_y>0 \}$

d) *Frequency Moment*: Given k, output $F_k = \sum_y C_y^k = N$



Random Approximate Counting

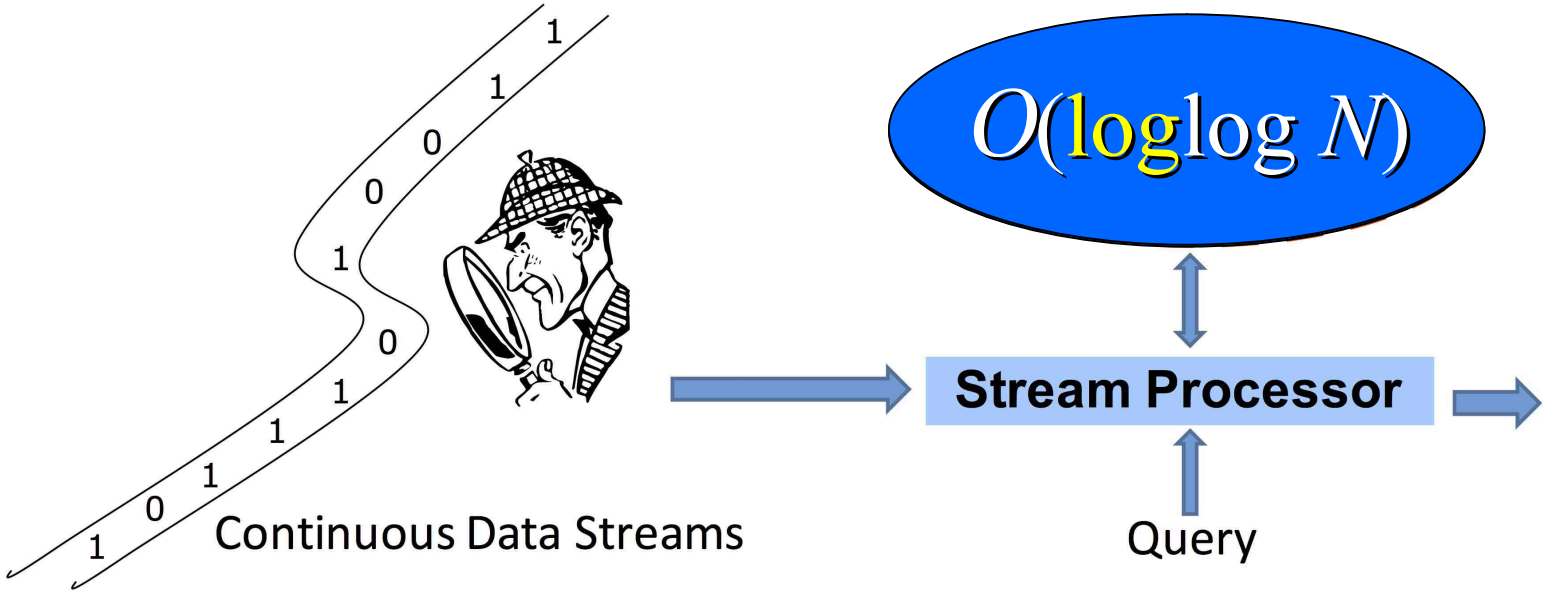
Example: Input sequence $x_n \in \{0,\ldots,M\text{-}1\}$, $n=0..N\text{-}1$

a) Count: Given $y \in \{0,\ldots,M\text{-}1\}$, output $\#\{n: x_n=y\} =: C_y$

Idea: Store and maintain $c_y := \log C_y$ instead of C_y

Increment c_y with every $2^{c_y} = C_y$-th occurrence of y,

in expectation: Flip c_y coins; if all *Heads*, then $++c_y$



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QUIZ

Let A denote a $d \times d$ matrix.

Prove: a) $\max |A^2| \leq d \cdot (\max |A|)^2$

Prove: b) $\max |A^K| \leq d^{K-1} \cdot (\max |A|)^K$

$$\max |A^2| \leq d \cdot (\max |A|)^2, \quad \max |A^K| \leq d^{K-1} \cdot (\max |A|)^K$$