Martin Ziegler *Total 20 pts + 5 bonus pts*

Donghyun Lim, Dongseong Seon, Ivan Koswara

# CS204

## Spring 2018, Homework #2

**Problem 1.** *1 + 1 + 1 + 1 + 1 + 1 pts*

Classify each example below as one of the following types; no justification needed.

**Types:** Theorem, Lemma, Corollary, Conjecture, Counterexample, Trivial proof, Vacuous proof, Proof by cases, Proof by contraposition, Proof by contradiction, Proof by induction

a) The famous proven statement "if $n \geq 3$ is an integer, then $a^n + b^n = c^n$ has no positive integer solutions $a, b, c$".
b) In particular, the statement "$a^3 + b^3 = c^3$ has no positive integer solutions $a, b, c$".
c) The number 2 to the claim "all primes are odd".
d) The following proof of the statement "all primes greater than 2 are odd": if $n > 2$ is even, then $n = 2k$ where $k > 1$ and so $n$ is not prime.
e) The famous unproven statement "every even integer $\geq 4$ is the sum of two primes".
f) The following proof of the statement "every even integer $\geq 4$ is the sum of two integers": every integer $n$ is the sum of two integers $n$ and 0.

**Problem 2.** *3 + 2 pts*

The following theorem is true, but the given proof is incorrect.

**Theorem.** If $x$ is a real number satisfying $(x^2 - x + 1)^2 - (x^2 - x + 1) + 1 = 1$, then $x$ also satisfies $x^2 - x + 1 = 1$.

**Proof.** Since $x^2 - x + 1 = 1$, plugging it into the original equation gives $1^2 - 1 + 1$ on the left hand side, which is equal to 1, the right hand side. So the claim is proven.

a) What is wrong with the above proof?
b) (Bonus) Give a correct proof of the theorem.

**Problem 3.** *2 + 2 + 2 + 1 + 2 + 2 + 2 + 1 pts*

You will be asked to prove the following theorem. Follow the given outline. If you cannot solve an item, you may still use it in the next steps. Recall that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

**Theorem.** Suppose $S \subseteq \mathbb{N}$ has at least 3 members and satisfies the following property: for all $n \in \mathbb{N}$, if $n \geq 2$ and $n$ is in $S$, then both $n + 3$ and $n - 2$ are also in $S$. Then $S = \mathbb{N}$.

a) Write the property "for all $n \in \mathbb{N}$, if $n \geq 2$ and $n$ is in $S$, then both $n + 3$ and $n - 2$ are also in $S$" in first-order logic.
b) Prove that $S$ has an element $b \geq 2$, by contradiction. (Hint: Use the fact that $S$ has at least 3 members.)
c) Using the $b$ from item b), prove that $S$ has an element $c \geq 2$ that is even, by cases. (Hint: Consider the parity of $b$.)
d) We will now prove that $S$ contains all elements $n \geq 2$, by induction. Write down the proposition $P(n)$ to be proved by induction.
e) Prove the basis step $P(2)$. (Hint: Use $c$ from item c).)
f) Prove the induction step $P(n) \to P(n + 1)$. (Hint: Use the property of $S$.)
g) (Bonus) This proof is not complete yet. What is missing? Complete the proof.
h) (Bonus) What happens if $|S| \geq 2$ instead of $|S| \geq 3$? Can we still prove $S = \mathbb{N}$?