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CS204
Fall 2017, Assignment #3

Problem 1. *1 + 1 + 2 pts*

Let $R = \{(1,1),(1,3),(1,4),(2,2),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4),(4,5),(5,3),(5,5)\}$. Is R

a) reflexive?
b) symmetric?
c) transitive?

If it is, just say yes; otherwise give a counterexample.

Problem 2. *2 + 4 + 3 pts*

For any given binary relation $R, S \subseteq A \times A$, define $R^n := \begin{cases} \{(a,a)|a \in A\}(= I_A) & n = 0 \\ R & n = 1 \\ R^{n-1} \circ R & n \geq 2 \end{cases}$

where $R \circ S = \{(a,c) \in A \times A | \exists c \in A : (a,c) \in R \wedge (c,b) \in S\}$.
Also define $R^{-1} := \{(b,a)|(a,b) \in R\}$.
Then define the **reflexive closure of** R as $r(R) := R \cup I_A$, the **symmetric closure of** R as $s(R) := R \cup R^{-1}$, and the **transitive closure of** R as $t(R) := \cup_{n=1}^{\infty} R^n$.
For any given binary relation $R \subseteq A \times A$:

a) Prove that the reflexive closure $r(R)$ is reflexive; and remains symmetric/transitive if R was. Also prove that every reflexive relation containing R also contains $r(R)$ as subset.
b) Prove that the symmetric closure $s(R)$ is symmetric; and remains reflexive if R was. Give a counter example where transitivity is not preserved.
c) Prove that the transitive closure $t(R)$ is transitive; and remains reflexive/symmetric if R was. *Hint: Think about each* $R^n (n \geq 1)$.

Problem 3. *3 + 4 pts*

Let $A_n := \{f | f : \{0,1\}^n \to \{0,1\}\}$.

a) Define a relation $R_n := (A_n, \preceq)$ s.t. given $f, g \in A_n, f \preceq g$ if and only if for every possible input $(a_1, \ldots, a_n) \in \{0,1\}^n$, if $f(a_1, \ldots, a_n) = 1$ then $g(a_1, \ldots, a_n) = 1$. It can be represented as a logical expression: $(f, g) \in R \leftrightarrow (?)$. Write ? as a logical expression using appropriate quantifiers. Feel free to use the definition of the **preimage**.
b) Show that R_n is a partially ordered set.

Problem 4. *2 + 3 pts*

a) Construct a set and an ordering that has *no* minimal element.
b) Construct a set and an ordering having *two* minimal elements and *no* least element.

For each item, also show why the set you constructed satisfies the given condition.