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CS204
Fall 2017, Assignment #5

Problem 1. *1 + 1 + 1 + 2 + 2 pts*

How many ways are there, when picking 5 cards (without order) out of a deck of 52, of getting

a) three of a kind,
b) full house,
c) two pair,
d) straight,
e) flush?

Write your calculation process.

Problem 2. *1 + 2 + 2 + 3 + 3 pts*

Suppose that you are a TA of discrete mathematics course and you have to distribute 11 points among 5 problems. Answer how many ways are there and explain why for each cases.

a) If you can distribute 0 points on any problems.
b) If every problem must receive at least 1 point.
c) If in addition to (b), at least one problem should receieve at least 3 points.
d) If in addition to (b), number of points must not decrease from problem to problem. (For example, distributing points like 1 + 2 + 3 + 2 + 3 is not allowed.)
e) If two problems should have even points(possibly zero) and other three problems should have odd points.

Problem 3. *4 + 5 + 2 + 5 pts*

Let $A_n := \{f | f : \{0,1\}^n \to \{0,1\}\}$, and let R_n be the relation that we defined in Problem 3 of Assignment 3, which is a partially ordered set as you proved.
From R_n, prohibit reflexivity and transitivity to define a new relation S_n: that is, for any $f, h \in A_n$, $(f,h) \in S_n$ if and only if the below two arguments holds:

(i) $f^{-1}(1) \subsetneq h^{-1}(1)$,
(ii) $\nexists g \in A_n$ such that $\{f^{-1}(1) \subsetneq g^{-1}(1)\} \wedge \{g^{-1}(1) \subsetneq h^{-1}(1)\}$.

Using R_n and S_n, answer the following:

a) Let $R_{n,g} := \{f | (f,g) \in R_n\}$. By assuming $|g^{-1}(1)| = m$, $|R_{n,g}| = 2^m$ for any $0 \le m \le 2^n$. Explain why it holds, and use it to calculate $|R_n|$.
b) Let $G_{n,m} := \{f | f \in A_n, |f^{-1}(1)| = m\}$ for $0 \le m \le 2^n$. For $1 \le m \le 2^n$, represent $C_{n,m} := |(G_{n,m-1} \times G_{n,m}) \cap S_n|$ as a function of n and m. You have to show how to count the elements of $(G_{n,m-1} \times G_{n,m}) \cap S_n$ to get a full credit.
c) Calculate $C_{2,m}$ for all $1 \le m \le 4$ to obtain what is $|S_2|$.
d) Let $N := 2^n$. Prove that $|S_n| = \frac{N2^N}{2}$ using the result of b) and properties of binomial coefficient.

Problem 4. *3 + 1 + 4 pts*

Suppose that there are n coins and k persons. ith person takes c_i coins, but unfortunately, whenever $i < j$, ith person cannot take coins more than jth person takes. Some people even may not take any coins. After they took coins, no coins are remaining. (For example, if there were 10 coins and 3 people, $(c_1, c_2, c_3) = (2, 4, 4)$ is valid. But $(c_1, c_2, c_3) = (2, 5, 3)$ or $(2, 3, 3)$ is invalid.

a) Let $D_{n,k}$ be the number of ways to take coins. Represent $D_{n,k+1}$ as a recurrence relation using $D_{?,k}$ family.
b) Calculate $D_{n,2}$ and explain why.
c) Calculate $D_{n,3}$ and explain why. Don't use **ceiling** nor **floor** functions in your final answer. Instead, divide the case to represent you answer for each n.

Problem 5. *1 + 3 + 2 + 2 pts*

Recall the Karatsuba's algorithm from the lecture: Consider two polynomials of degree n, $A(x) := A_0(x) + A_1(x)x^{\frac{n}{2}}$ and $B(x) := B_0(x) + B_1(x)x^{\frac{n}{2}}$. When you use multiplication 3 times calculating $c_0(x) := A_0(x)B_0(x)$, $c_1(x) := (A_0(x)+A_1(x))(B_0(x)+B_1(x))$ and $c_2(x) := A_1(x)B_1(x)$, you can calculate each coefficient of $A(x)B(x) := C_0(x) + C_1(x)x^{\frac{n}{2}} + C_2(x)x^n$ only by linear combination of $c_0(x)$, $c_1(x)$, $c_2(x)$.

a) Note that $A(x)B(x) = (A_0(x)+A_1(x)x^{\frac{n}{2}})(B_0(x)+B_1(x)x^{\frac{n}{2}}) = A_0(x)B_0(x)+(A_1(x)B_0(x)+A_0(x)B_1(x))x^{\frac{n}{2}} + A_1(x)B_1(x)x^n$. Thus it is easy to see that $C_0(x) = c_0(x)$ and $C_2(x) = c_2(x)$. Verify that $(A_1(x)B_0(x)+A_0(x)B_1(x)) = C_1(x) = c_1(x)-c_0(x)-c_2(x)$.
b) Now let's divide $A(x)$ and $B(x)$ into three pieces. Also for the sake of simplicity, let $n = 3k$. Redefine $A(x) := A_0(x)+A_1(x)x^k+A_2(x)x^{2k}$ and $B(x) := B_0(x)+B_1(x)x^k+B_2(x)x^{2k}$. Let $A(x)B(x) = C_0(x) + C_1(x)x^k + \cdots C_4(x)x^{4k}$ and

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\begin{aligned}
c_0(x) &= A_0(x)B_0(x) \\
c_1(x) &= (A_0(x) + A_1(x) + A_2(x))(B_0(x) + B_1(x) + B_2(x)) \\
c_{-1}(x) &= (A_0(x) - A_1(x) + A_2(x))(B_0(x) - B_1(x) + B_2(x)) \\
c_{-2}(x) &= (A_0(x) - 2A_1(x) + 4A_2(x))(B_0(x) - 2B_1(x) + 4B_2(x)) \\
c_\infty(x) &= A_2(x)B_2(x)
\end{aligned}
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Verify that $A_1(x)B_0(x)+A_0(x)B_1(x) = C_1(x) = \frac{1}{2}c_0(x)+\frac{1}{3}c_1(x)-c_{-1}(x)+\frac{1}{6}c_{-2}(x)-2c_\infty(x)$.
c) Similar to (b), other coefficients, such as $C_0(x), C_2(x), \cdots$ can be expressed as a linear combination of $c_{-2}, \cdots, c_1, c_\infty$. Embrace that fact and derive a recurrence relation on $f(n)$, which denotes the number of operations to calculate $A(x)B(x)$ in this new method.
d) Solve the recurrence relation in (c).