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CS204
Fall 2017, Assignment #7

Problem 1. *5 pts*

Use mathematical induction to prove the following item:

Suppose that there is a real sequence $\{a_n\}$ satisfying $a_{i+j} \le a_i + a_j$ for all $i, j \in \mathbb{Z}^+$. Prove that $\sum_{i=1}^{n} \frac{a_i}{i} \ge a_n$ for all $n \in \mathbb{Z}^+$.
Hint: Use $a_1 + \cdots + a_n$ in your proof.

Problem 2. *3 + 6 pts*

The given function $M : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined using the rule

$$M(n) := \begin{cases} n - 10 & (n \ge 101) \\ M(M(n+11)) & (n \le 100) \end{cases}$$

for all positive integers n.

(a) Calculate $M(107), M(101), M(89), M(78)$.

(b) Write and use a computer program (Language: C/C++/Python) to **verify** that M is totally defined, and list its values for all $n \le 100$. Verify in this problem means: show that your program which simulates given recursive function not only always terminates but also return a positive integer always. You should print or attach your source code in your hard-copy submission as well as your result.

Problem 3. *1 + 3 + 2 pts*

Recall the definition of the fibonacci number:

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \forall n \in \mathbb{N}.$$

In the above case, we calculated F_n only for non-negative integers n. But, why not for negative integers?
Move F_{n+1} to the left side from the given recurrence relation to obtain $f_n = F_{n+2} - F_{n+1}$ for all $n \in \mathbb{Z} \setminus \mathbb{N}$.

(a) Manually calculate F_n for $n = -1, -2, -3, -4$.

(a) Prove that $F_n = \frac{1}{\sqrt{5}}\left((\frac{\sqrt{5}-1}{2})^{-n} - (-\frac{\sqrt{5}+1}{2})^{-n}\right)$ for $n \in \mathbb{Z} \setminus \mathbb{N}$ by induction.

(b) Since $\mathbb{Z}$ has no least element, it seems that we cannot use the induction to calculate the whole sequence $\{F_n\}_{n \in \mathbb{Z}}$. However, we did. Then what trick is used here? Answer according to the domain we concerned.

Problem 4. *5 pts*

There is a weird recursively defined sequence called Hofstadter Sequence. It is defined as:

$$F(n) = \begin{cases} 1 & (n = 0) \\ n - M(F(n-1)) & (n \ge 1) \end{cases}, \quad M(n) = \begin{cases} 0 & (n = 0) \\ n - F(M(n-1)) & (n \ge 1) \end{cases}.$$

For small n, $F(n)$ differs from $M(n)$ frequently, but the case $F(n) \ne M(n)$ occurs less as n gets bigger. Find first eight integers $n_1 < \cdots < n_8$ s.t. $F(n_i) \ne M(n_i)$.