Martin Ziegler, Svetlana Selivanova

Dongseong Seon, Hyunwoo Lee, Ivan Koswara, Seungwoo Schin

Total 10 pts

CS204

Fall 2018, Homework #8

Problem 1. *2+2+3+3 pts*

A $2 \times n$ board will be tiled with 2×1 dominoes and 2×2 squares so there is no overlap and the board is fully covered, without any part of a tile sticking out of the board. Dominoes may be rotated. Tiles of the same size are indistinguishable. Let a_n be the number of ways to tile a $2 \times n$ board this way.

a) Compute a_1, a_2, a_3, a_4 by listing all possible tilings.
b) Derive a recurrence relation for a_n and explain why it is correct. Verify that the relation holds for $n = 3, 4$. (The value of a_n should only depend on a_{n-1} and a_{n-2}.)
c) Solve the recurrence relation. Simplify it as much as you can.
d) Solve the following unrelated recurrence relation: $b_1 = 3, b_2 = 5$, and for $n \geq 3$,

$$b_n = b_{n-1} + 2b_{n-2} + 9n2^n.$$

Problem 2. *2+2+3+3 pts*

Let p, q, r be positive real numbers. Consider the following recurrence relation in t_k, where $t_0 = q$ and for all $k \geq 1$,

$$t_k = p \cdot t_{k-1} + q \cdot r^k.$$

Important: You might need to divide your answer into several cases depending on the values of p, q, r (similar to how Master's Theorem has three cases).

a) Find a particular solution for t_k. (You should have two cases.)
b) Solve the recurrence relation. (You should have two cases.)
c) Derive the asymptotic growth of t_k in Big-Θ notation. (You should have three cases.)
d) Prove Master's Theorem as per the slides using the result in item (c).

Problem 3. *5+5 pts*

You have learned one form of Principle of Inclusion-Exclusion. This is another form.

Principle of Inclusion-Exclusion. Let U be the universe set. Let $P_1, P_2, \ldots, P_n$ be functions from U to $\{T, F\}$. Define $A = \{x | x \in U \wedge \forall i\, (P_i(x) = F)\}$. Also, for each $S \subseteq \{1, 2, \ldots, n\}$, define $f(S) = \{x | x \in U \wedge \forall i \in S\, (P_i(x) = T)\}$. Then

$$|A| = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} |f(S)|.$$

(Intuitively, the P_i's are "bad" properties that may apply to elements in U. $f(S)$ is the set of elements that have bad properties P_i for all $i \in S$ (and possibly other bad properties). A is the set of "good" elements, that don't have any bad property.)

a) Prove the above version of Principle of Inclusion-Exclusion. You may cite the version you learned in the lecture.
b) Prove the following identity using the above version of Principle of Inclusion-Exclusion:

$$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n.$$

Hint: Count surjections from $\{1, 2, \ldots, n\}$ to itself.