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CS204

Fall 2018, Homework #10

A **path** from u to v is a sequence $(v_0, v_1, v_2, \ldots, v_n)$ of *distinct* vertices such that $v_0 = u, v_n = v$, and (v_{i-1}, v_i) is an edge for all i. If the graph is directed, then (v_{i-1}, v_i) must be a directed edge from v_{i-1} to v_i. The **length** of the path is the value of n; this may be 0. A path is **Hamiltonian** if in addition it uses all vertices of the graph.

Problem 1. *2+2+3+3 pts*

Let m, n be positive integers. An $m \times n$ **grid graph** is a simple, undirected graph that consists of vertices labeled (x, y) where $1 \le x \le m, 1 \le y \le n$, and two vertices $(x_1, y_1), (x_2, y_2)$ are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$.

a) Draw the 3×4 grid graph. (You don't need to label the vertices.)
b) Derive a formula for the number of edges of $m \times n$ grid graph and prove it.
c) Prove all grid graphs are bipartite.
d) Prove all grid graphs have a Hamiltonian path.

Problem 2. *2+4+4 pts*

Let G be an undirected multigraph (it may have multiple edges between vertices, but no self-loops). Label the vertices of G with $1, 2, 3, \ldots, n$, and let d_i be the degree of vertex i. The **degree sequence** of G is the sequence $(d_1, d_2, d_3, \ldots, d_n)$, where the labels of the vertices are chosen so that $d_1 \ge d_2 \ge d_3 \ge \ldots \ge d_n$.

a) Write down the degree sequence of a 3×4 grid graph (see Problem 1a).
b) Let S be the sum of the elements of a degree sequence. Prove that S is even and $d_i \le S/2$ for all i.
c) Let $(d_1, d_2, d_3, \ldots, d_n)$ be a sequence of non-negative integers such that it satisfies the properties in item (b) and $d_1 \ge d_2 \ge d_3 \ge \ldots \ge d_n$. Prove that $(d_1, d_2, d_3, \ldots, d_n)$ is the degree sequence of some multigraph.

Problem 3. *2+3+2+3 pts*

A **tournament** is a simple, directed graph where between every pair of vertices there is exactly one edge, directed one way or another. A **transitive** tournament additionally satisfies the following property: whenever there are two edges directed $a \to b$ and $b \to c$, the edge between a, c is directed $a \to c$. (Recall the transitivity property of a relation.)

a) Prove all tournaments have a Hamiltonian path. (**Hint:** Induction on number of vertices.)
b) Prove that in every tournament, there exists a vertex u satisfying the following: for every vertex v, there exists a path from u to v with length at most 2. (**Hint:** Consider a vertex with the highest outdegree.)
c) Prove that a transitive tournament doesn't have any cycle. (**Hint:** Induction on cycle length.)
d) Prove that in a tournament with 2^n vertices, you can find an induced subgraph of $n + 1$ vertices that is a transitive tournament. (**Hint:** Induction on n.)