Martin Ziegler

Donghyun Lim, Dongseong Seon, Ivan Koswara

Total 25 pts + 15 bonus pts

CS204

Spring 2018, Homework #6

Problem 1. *7 + bonus 3 pts*

Each of the following counting problem is worth 1 point. The answers are given below, but in jumbled order, and some answers are extraneous. You only need to give the answer, but you may give a sketch of your method for partial points. Problems range in difficulty and are *not* sorted; part of learning is to be able to judge which problems are doable.

Answers: 15, 20, 21, 24, 30, 32, 35, 36, 40, 44, 48, 60, 64, 70, 75, 81, 90, 96, 100, 120

a) Number of ways to give 4 indistinguishable candies to 4 students.
b) Number of bitstrings of length 6 that have an even number of 0's.
c) Number of functions from $\{1,2,3,4\}$ to $\{1,2,3\}$ that are surjective.
d) Number of ways to divide 6 students into 3 indistinguishable teams of 2.
e) Number of subsets of $\{1,2,3,4,5,6\}$ that don't contain consecutive numbers.
f) Number of ways to permute $(1,2,3,4,5)$ so no element ends up in its original position.
g) Number of 2-digit numbers. (Numbers are written in decimal and may not begin with digit 0.)
h) Number of ways to place 3 rooks on a 4×4 chessboard so no two rooks are in the same row/column.
i) Number of ways to move from $(0,0)$ to $(4,4)$ in eight steps, where for each step, from (x,y) you can only go right to $(x+1,y)$ or go up to $(x,y+1)$.
j) Number of ways to color the sides of a cube with 6 colors so the sides get different colors. (If a coloring can be rotated to another, they are the same.)

Problem 2. *3 + 3 + 3 pts*

Prove the following theorems. Proofs are expected to use the Pigeonhole Principle; other kinds of proofs can get a maximum of 2.5 points.

a) Among any 10 points in the unit square, there exist two points with distance < 0.5.
b) For any positive integer n, among any $n+1$ numbers from the set $\{1,2,3,\ldots,2n\}$, there exist two numbers a,b where b is a multiple of a.
c) There are $n \geq 2$ people on Facebook. Some pairs of people are related by a friendship relation; the friendship relation is symmetric and irreflexive. Then there exist a pair of people that have the same number of friends.

Problem 3. *3 + 3 + 3 pts*

Prove the following identities for all natural numbers $n, r \geq 0$. Combinatorial proofs (bijection or double counting) are expected; other kinds of proof can get a maximum of 2.5 points.

a) $\displaystyle\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$ b) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ c) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Problem 4. *bonus 3 + 3 + 3 + 3 pts*

All of these problems are optional, but they are also challenging. Good luck.

a) Generalize Problem 1a to dividing kn students into n indistinguishable teams of k. Determine the number of ways to do so. Simplify your final expression as much as you can.
b) There are $n \geq 2$ people on Facebook (see Problem 2c). Prove that there are an even number of people that have an odd number of friends.
c) $n \geq 0$ is a natural number. Evaluate $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}$ with binomial theorem, then prove the identity combinatorially by comparing the terms with even k and the terms with odd k.
d) There are 6 people on Facebook (see Problem 2c). Prove that there exist three people that are either all friends with each other or none is friends with each other.