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Total 17 pts + 8 bonus pts

CS204
Spring 2018, Homework #7

Problem 1. *(2 + bonus 1) × (1 + 1 + 2) pts*

Consider the following recurrence relations.

1) $a_n = a_{n-1} + 2$ 2) $a_n = a_{n-1} + 2a_{n-2}$ 3) $a_n = 2a_{n-1} - a_{n-2}$ 4) $a_n = 2a_{n-1} + 1$

Each of the following counting problems can be solved using recurrence. If we denote a_n to be the answer to the problem, we can write down a recurrence relation for a_n, and it is one of the above. However, the relations are not given in any specific order, and one relation is extraneous.

i) Number of ways to divide a $2 \times n$ rectangle into blocks of size 1×2, 2×1, and/or 2×2.
ii) Number of ways to choose a non-empty subset from a set of size n.
*iii) Number of strings of n bits that don't contain 10.

For each counting problem, answer the following.

a) Choose the relation that describes it, and give the initial conditions.
b) Explain why the relation you choose is correct, by giving a double counting proof that both sides of the recurrence relation are equal.
c) Solve the chosen recurrence relation by giving an explicit formula (not a recurrence) for a_n.

Problem 2. *2 + 1 + 2 + bonus 2 pts*

Solve the following problems related to recurrence relations. You need to show your work.

a) Suppose that $a_n = 3^{n-1}$ for $n = 1, 2, 3$, and $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3} - 2$ for $n \geq 4$. Find an explicit formula for a_n. (Hint: $a_n = n^2$ is a particular solution for the recurrence relation.)
b) Strassen's algorithm, used to multiply two $n \times n$ matrices, runs in time $f(n)$ satisfying the relation $f(n) = 7f(n/2) + 18n^2$. Use Master Theorem to obtain the asymptotic growth of $f(n)$.
c) A string of bits is called *social* if every bit has at least one neighbor that is equal to itself. For example, 00111 is social, because every bit has a neighbor that is equal. But 00110 isn't, because the last 0 doesn't have a neighbor that is also 0. Construct a recurrence relation to count the number of social strings of length n, and give the initial conditions. (You don't need to solve it.)
*d) A function f from a set S to itself is called an *involution* if $f(f(x)) = x$ for all $x \in S$. For example, the additive inverse function $f(x) = -x$ on the integers $\mathbb{Z}$ is an involution. Construct a recurrence relation to count the number of involutions on the set $\{1, 2, \ldots, n\}$, and give the initial conditions. (You don't need to solve it.)

Problem 3. *2 + 2 + bonus 2 pts*

Solve the following problems related to Principle of Inclusion-Exclusion. You need to show your work.

a) Determine the number of positive integers ≤ 2018 that is **not** divisible by any of 2, 3, 4, 5, or 6.
b) A bijection π on $\{1, 2, \ldots, n\}$ is called a *derangement* if no element ends up in its original position. That is, $\pi(i) \neq i$ for all i. Suppose there are D_n derangements of n elements; show that

$$D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

*c) Prove the following identity: for all integer $n \geq 1$,

$$\sum_{k=1}^{n} (-1)^{n-k} k^n \binom{n}{k} = n!$$

(Hint: You can use double counting, where the summation uses Principle of Inclusion-Exclusion.)