Martin Ziegler *Total 12 pts*

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CS204

Spring 2018, Homework #9

Problem 1. *1+1+2+2 pts*

A *balanced parenthesis string*(*BPS* for short) is defined recursively as follows:

- () is a BPS.
- For any BPS w, (w) is a BPS.
- For any BPS w and v, wv is a BPS.
- Only those are BPSs.

Given a string w, a sequence M_w is defined in such a way that

$M_w[i] = \#$ of opening parentheses in $w[1 \cdots i] - \#$ of closing parentheses in $w[1 \cdots i]$.

For example,

$$M_{()(())} = [1, 0, 1, 2, 1, 0].$$

Note that index starts from 1. Also note that M_w and w are of the same length.
Given a BPS w of length n, show that the followings hold, by *structural induction.* You have to prove them by structural induction to receive points.

(a) $M_w[1] = 1$
(b) $M_w[n] = 0$
(c) $M_w[i] = M_w[i-1] + 1$ or $M_w[i] = M_w[i-1] - 1$ for $i = 2 \cdots n$.
(d) $M_w[i] \geq 0$ for $i = 1 \cdots n$

Problem 2. *3+1+2 pts*

A *mountain sequence* m is a sequence of length $n \in \mathbb{Z}^+$ satisfying:

- $m[1] = 1$
- $m[n] = 0$
- $m[i] = m[i-1] + 1$ or $m[i] = m[i-1] - 1$ for $i = 2 \cdots n$.
- $m[i] \geq 0$ for $i = 1 \cdots n$

For example, $[1, 2, 3, 2, 1, 2, 1, 0]$ is a mountain sequence, and $[1, 0, -1, 0]$ is not. It kind of looks like mountains if you draw the graph, hence the name *mountain sequence.*

(a) Let w be a nonempty string consisting only of opening and closing parentheses. Prove that if M_w is a mountain sequence, then w is a BPS.
(b) Write an efficient algorithm in pseudo code that, given a mountain sequence m, returns a string w satisfying $M_w = m$.
(c) Fix a positive even number n. Let $\mathbb{B}$ denotes the set of BPSs of length n, and $\mathbb{M}$ denotes the set of mountain sequences of length n. Your algorithm in (b) can be considered as a mapping $\mathbb{M} \to \mathbb{B}$. Prove that your algorithm (b) is a well-defined mapping (i.e., its codomain is actually $\mathbb{B}$) and that it is injective.