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Total 17 pts + 8 bonus pts

CS204
Spring 2018, Homework #10

Problem 1. *7 × 1 pts*

In this problem, all graphs are **simple and undirected**.

If G is a graph, define the **complement** of G, called $\overline{G}$, to have the same vertex set as G, and u, v are adjacent in $\overline{G}$ if and only if they are **not** adjacent in G.

For each subproblem, you are given a specific set of conditions. Your task is to **construct a graph satisfying those conditions**. You do **not** have to prove that your graph is correct, but there is also no partial credit.

a) Graph with 4 vertices that **not** bipartite, but deleting any edge makes it bipartite.
b) Graph with 4 vertices that is isomorphic to its complement.
c) Graph with 5 vertices and 6 edges, and there exist a pair of vertices with distance exactly 3.
d) Graph with 5 vertices and 6 edges, and it doesn't have three vertices all adjacent to each other.
e) Graph with 6 vertices and 5 edges, and its complement is **not** connected.
f) Graph with 6 vertices that is connected and Eulerian, and its complement is Hamiltonian.
g) Graph with 7 vertices, and every pair of vertices have exactly one common neighbor.

Problem 2. *1 + 1 + 1 + 2 + bonus 2 + 2 pts*

In this problem, all graphs are **simple and undirected**.

A k**-coloring** of G is a function $c : V(G) \to \{1, 2, \dots, k\}$ such that $c(u) \neq c(v)$ whenever u, v are adjacent. Intuitively, we color the vertices of G with k colors so no two adjacent vertices are colored the same color. The **chromatic number** $\chi(G)$ of G is the smallest positive integer k such that G has a k-coloring. The chromatic number of a graph is known as something difficult to compute exactly.

a) Look up a 10-vertex graph called the **Petersen graph** P, and draw it.
b) Find a 3-coloring of P. Label each vertex in your drawing with the corresponding color. (You don't need actual colors; labels of 1, 2, 3 are enough.)
c) Recall that C_n is the cycle graph on n vertices. Prove that if $n \geq 3$ is odd then $\chi(C_n) = 3$.
d) Prove that if G is a graph and H is a subgraph of G, then $\chi(H) \leq \chi(G)$.
*e) Prove that P doesn't have a 2-coloring. Conclude that $\chi(P) = 3$. (Hint: Use parts c and d.)
*f) Suppose $\Delta(G)$ is the maximum degree among the vertices of G. Prove that $\chi(G) \leq \Delta(G) + 1$. (Hint: Induct on the number of vertices.)

Problem 3. *2 + 1 + 2 + bonus 2 + 2 pts*

Recall that a **walk** in a directed graph G is a sequence $(v_0, v_1, v_2, \dots, v_k)$ of vertices in G such that $v_{i-1} \to v_i$ is an edge for all $i = 1, 2, \dots, k$; we say k is the length of the walk. It is a **path**, if additionally, all $v_0, v_1, v_2, \dots, v_k$ are distinct. A **Hamiltonian path** is a path that uses all vertices. A vertex v is **reachable** from another vertex u if there is a path from u to v.

A **tournament** is a directed graph such that there exists exactly one edge between every pair of vertices. (The edge can be directed in one way or another, but not both.) It is called a tournament because we can imagine the vertices are players participating in a round-robin tournament, where each pair of players play against each other; an edge points from the winner to the loser.

a) Draw two non-isomorphic tournaments on 4 vertices and prove that they are not isomorphic.
b) Draw a Hamiltonian path on each tournament you drew in part a.
c) Prove that every tournament has a Hamiltonian path. (Hint: Induct on the number of vertices.)
*d) Prove that every tournament has a vertex u that can reach all other vertices. (Hint: Take a vertex that can reach the most number of vertices, and prove this works.)
*e) Improve part d. Prove that every tournament has a vertex u that can reach all other vertices **with a path of length at most 2**.