CS204 Discrete Mathematics, Spring 2018

Recitation #3 Solution

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1. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive by definition.

a) $\{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$

b) $\{(2,4),(4,2)\}$

c) $\{(1,2),(2,3),(3,4)\}$

d) $\{(1,1),(2,2),(3,3),(4,4)\}$

Solution.

a) Let the relation be R

$(1,1) \notin R \Rightarrow$ not reflexive.

$(4,2) \notin R \Rightarrow$ not symmetric

$(3,2) \in R \wedge (2,3) \in R \wedge 2 \neq 3 \Rightarrow$ not antisymmetric

It is transitive. It is enough to show that for x, y s.t. $(x,y) \in R \wedge (y,z) \in R,\ (x,z) \in R$.

i) $(x,y) = (2,2) or (3,3)$: $x = y \Rightarrow (x,z) = (y,z) \in R$.

ii) $(x,y) = (2,3)$: $(y,z) = (3,4) or (3,3) \Rightarrow (x,z) = (2,4) or (2,3) \in R$

iii) $(x,y) = (2,4)$: there's no (y,z) $s.t. y = 4\ and\ (y,z) \in R$

iv) $(x,y) = (3,2)$: $(y,z) = (2,2) or (2,3) \text{or} (2,4) \Rightarrow (x,z) = (3,2) or (3,3) or (3,4) \in R$

v) $(x,y) = (3,4)$: there's no (y,z) $s.t. y = 4\ and\ (y,z) \in R$

b) symmetric

c) antisymmetric

d) reflexive, symmetric, antisymmetric, transitive

2. Check properties for problem 1c and 1d again by representing them by matrices.

Solution.

When we represent c) to matrix, we get

0	1	0	0
0	0	1	0
0	0	0	1
0	0	0	0

It has no 1 on diagonal entries. So it is not reflexive.

In order to be symmetric, it needs 1 on the symmetric entry of entry which has 1 as element(colored blue). So it's not symmetric.

If every symmetric entry of entry which has 1 has element 0, it is antisymmetric. So it's antisymmetric.

When we represent d) to matrix, we get identity matrix. This shows it is reflexive, symmetric, antisymmetric and transitive.

3. Check properties for problem 1a and 1b again by representing them by graphs.

4. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive,

where $(x,y) \in R$ if and only if

a) $xy \geq 1$.

b) $x = y + 1$ or $x = y - 1$.

c) $x \equiv y \pmod{7}$.

d) $x = y^2$.

Solution.

a) $x^2 < 1$ is possible if $x = 0$ so not reflexive.

$xy \geq 1 \Rightarrow yx \leq 1$ so it is symmetric.

$(1,2), (2,1)$ both in $R \Rightarrow$ it is not antisymmetric.

$$xy \geq 1 \Rightarrow x \neq 0 \wedge y \neq 0 \wedge \text{sgn}(x) = \text{sgn}(y)$$

where $\text{sgn}(x)$ denotes the sign of x.

$$yz \geq 1 \Rightarrow y \neq 0 \wedge z \neq 0 \wedge \text{sgn}(y) = \text{sgn}(z)$$

Thus

$$xy \geq 1 \wedge yz \geq 1 \Rightarrow x \neq 0 \wedge y \neq 0 \wedge z \neq 0 \wedge \text{sgn}(x) = \text{sgn}(y) = sgn(z) \Rightarrow xz \geq 1$$

b) $x = y + 1 \vee x = y - 1 \Leftrightarrow |x - y| = 1$

not reflexive. Symmetric. Not antisymmetric. Not transitive because $x = 0, y = 1, z = 2 \Rightarrow |x - y| = 1, |y - z| = 1, |x - z| \neq 1$

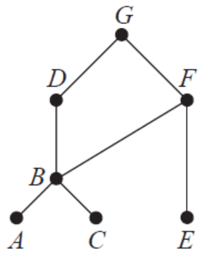
c) reflexive, symmetric, not antisymmetric, transitive

d) not reflexive; $(2,2) \notin R$
not symmetric; $(4,2) \in R, (2,4) \notin R$
antisymmetric; $\mathrm{x} = \mathrm{y}^2, \mathrm{y} = \mathrm{x}^2 \Rightarrow x = x^4 \Rightarrow x = 0 \;\; or \;\; 1.\;\; \mathrm{x} = 0 \Rightarrow \mathrm{y} = 0, \mathrm{x} = 1 \Rightarrow \mathrm{y} = 1.$
not transitive; $(4,2) \in R, (16,4) \in R, (16,2) \notin R$

5. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.
a) Show that R is an equivalence relation on A.
b) What are the equivalence classes of R?

Solution.
a) check reflexive, symmetric, transitive similar to problem 4.
b) $\{\ \{x \mid f(x) = y\} \mid y \in f(A)\}$ is the set of equivalent classes.
i.e. if $f(x) = y$ *then* $[x]_R = \{x \mid f(x) = y\}$

6. Answer these questions for the partial order represented by this Hasse diagram.



Solution.
a) Find the maximal elements. G
b) Find the minimal elements. A, C, E
c) Is there a greatest element? Yes: G
d) Is there a least element? No
greatest element means every other element is comparable with this element and is smaller than this element.
Maximal element means every other element is not comparable with this element or is smaller than this element.