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CS422

Spring 2019, Homework #3

Recall that $f \preccurlyeq g$ (f is "reducible" to g) means that a WHILE program with oracle g can compute f.

Problem 6 (5 points)

To the five constructs of a WHILE program, add the following command:

- $x_i \coloneqq \mathrm{RUN}(a, b, c)$

The semantic meaning of this command is: run the program encoded as a on input b for c steps. If it halts with output r, then set x_i to be $r + 1$; otherwise, set x_i to be 0. Using this command, write a program to solve the following item. You don't need to prove its correctness. You may use any convenience command that we have defined in the lecture (e.g. decrement), and you may define your own.

- Semi-decide the Nontriviality problem: On input x_1, terminate if the algorithm encoded as x_1 halts on *some* input, otherwise loop forever.

Problem 7 (5 + 5 + 5 points)

a. For a set S, define H^S to be the set of all program-input pairs that halt, where the program has access to oracle S. With this, specify H^H formally as a subset of $\mathbb{N}$.
b. Prove that $\mathbf{1}_{H^H}$ is not reducible to $\mathbf{1}_H$.
c. Prove that $\mathbf{1}_T$ is reducible to $\mathbf{1}_{H^H}$.

Problem 8 (5 + 5 + 5 points)

If $\varphi, \psi \colon \mathbb{N} \to \mathbb{N}$, define their disjoint sum $\varphi \oplus \psi$ as follows:

$$\varphi \oplus \psi(\langle x, w \rangle) = \begin{cases} \varphi(x) & \text{if } w \text{ is even} \\ \psi(x) & \text{if } w \text{ is odd} \end{cases}$$

a. Consider an oracle program that has access to two oracles φ, ψ instead of just one. Prove that we can modify this program to use only one oracle $\varphi \oplus \psi$.
b. Let $f \colon \mathbb{N} \to \mathbb{N}$ satisfies $\varphi \preccurlyeq f$ and $\psi \preccurlyeq f$. Prove that $\varphi \oplus \psi \preccurlyeq f$.
   Thus disjoint sum captures exactly the combined power of the base oracles and no more: any oracle f that is at least as powerful as both of the base oracles is also at least as powerful as their disjoint sum.
c. Let $\varphi', \psi' \colon \mathbb{N} \to \mathbb{N}$ satisfy $\varphi \preccurlyeq \varphi'$ and $\psi \preccurlyeq \psi'$. Prove that $\varphi \oplus \psi \preccurlyeq \varphi' \oplus \psi'$.

Problem 9 (10 bonus points)

The twin prime conjecture states that there are infinitely many pairs of distinct prime numbers p, q with $|p - q| \leq 2$; it is as of 2019 still not proven. Consider the decision problem PRIMEGAP: given input n, return "yes" if there are infinitely many pairs of distinct prime numbers p, q with $|p - q| \leq n$, otherwise return "no".
Prove that PRIMEGAP is decidable.