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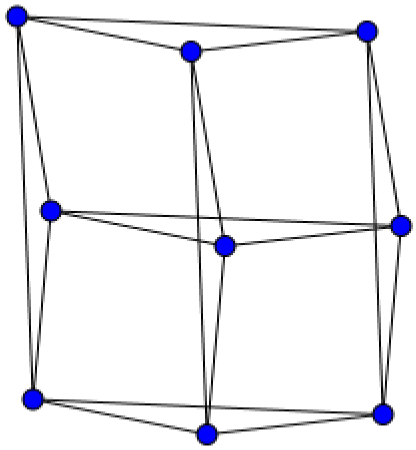
CS422

Spring 2019, Homework #4

Problem 10 ($7 \times 3 + 1$ points)



Let G be the 3×3 rook graph, pictured on the right. Answer the following questions about G and prove your answers.

a. Is G 3-colorable?
b. What is the size of the smallest edge cover of G?
c. What is the size of the smallest vertex cover of G?
d. Does G have an Eulerian cycle?
e. Does G have a Hamiltonian cycle?
f. What is the size of the largest clique in G?
g. What is the size of the largest independent set in G?
h. (Bonus) Do you know why it is called a rook graph? (Feel free to look this up.)

Problem 11 (5×3 points)

Consider the following Boolean formula φ:

$$\big(\neg(a \vee b \vee \neg c) \vee (a \wedge \neg b)\big)$$

Answer the following questions about φ and prove your answers.

a. Is φ in CNF? If not, convert it into an equivalent CNF formula.
b. Is φ in DNF? If not, convert it into an equivalent DNF formula.
c. Is φ satisfiable?
d. Is φ a tautology?
e. Does φ have an equivalent shorter formula? (The length of a formula is the number of times you need to invoke the construction rules. For example, the formula $(a \vee \neg b)$ has length 4: one to make a, one to make b, one to make $\neg b$, and one to make the entire formula.)

(Problems 12-13 on next page)

Problem 12 (3 + 3 + 6 + 6 points)

In a **Sudoku puzzle**, you are given an $k^2 \times k^2$ grid divided into k^2 boxes of size $k \times k$. Some cells have been filled with integers from the set $\{1, 2, \dots, k^2\}$. A **solution** to a Sudoku puzzle is a completely filled $k^2 \times k^2$ grid such that whenever there is a number in a cell of the puzzle, the same number is on the same cell in the solution, and every row, column, and box contains each of $1, 2, \dots, k^2$ exactly once. See the pictures on the right; the top one is a 4×4 Sudoku puzzle ($k = 2$) and the bottom one is its solution. You can verify that the solution is correct.

	1		
			2
1			
		3	

2	1	4	3
3	4	1	2
1	3	2	4
4	2	3	1

Define the decision problem SUDOKU as follows: given (the encoding of) a Sudoku puzzle, does it have a solution?
Define the decision problem SUDOKU_k as follows: given (the encoding of) a $k^2 \times k^2$ Sudoku puzzle, does it have a solution?

	4		
		2	
			2
1			

a. Does the Sudoku puzzle on the left belong to the problem SUDOKU? Prove it.
b. Suppose we change the 1 on the bottom-left cell into a 2. Does your answer for item a) change? Prove it.
c. Prove that SUDOKU is in **NP**.
d. For any fixed k, prove that SUDOKU_k is in **P**.

Problem 13 (6 + 6 points)

In this problem, you will see that the relationship between **P** and **NP** is very similar to the relationship between decidable and semidecidable problems: we can use a definition for **NP** and adjust it slightly to get an alternative definition for semidecidability.

Consider the set $L_p = \{x \mid \exists w.\, \ell(w) \leq \text{poly}(\ell(x)) \wedge \langle x, w\rangle \in L'_p\}$ where $L'_p \in \mathbf{P}$. This is the definition of a problem that is in **NP**: an input is in L_p if has a polynomial-length witness w that can be verified in polynomial time, where L'_p means "is w a correct witness for x?"

Now consider what happens if we get rid of all the polynomials (marked in dark red above): suppose L can be expressed as $\{x \mid \exists w.\, \langle x, w\rangle \in L'\}$ where L' is decidable.

a. Prove that L is semidecidable.
b. Conversely, prove that any semidecidable problem can be expressed in this form.