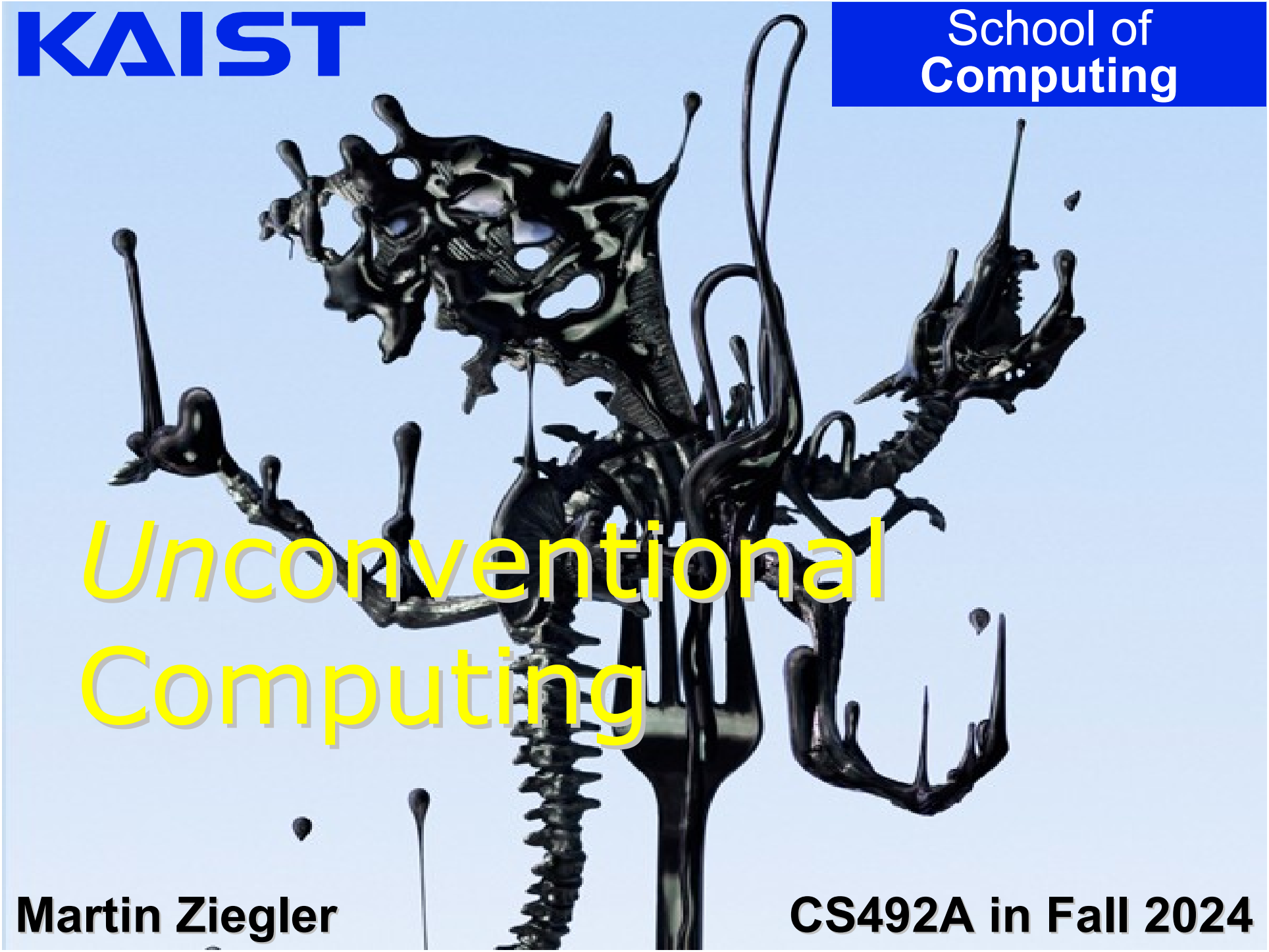
KAIST

School of **Computing**



*Un*conventional Computing

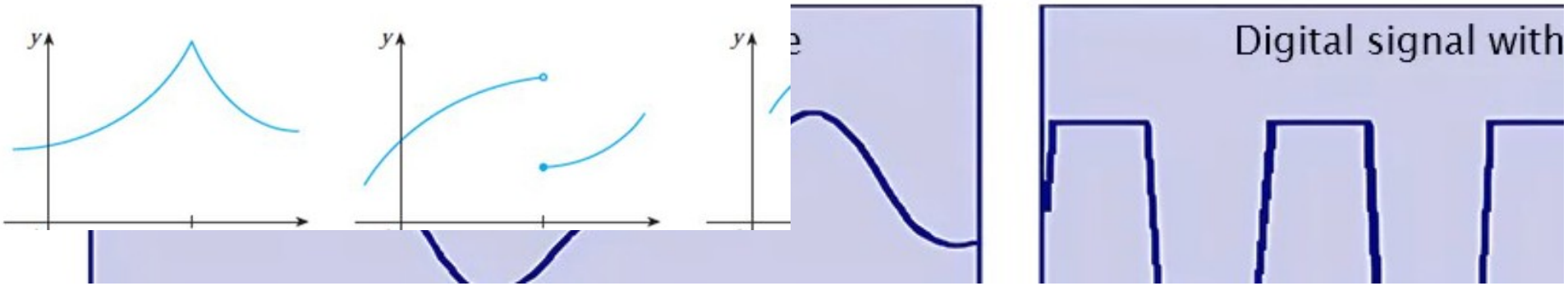
Martin Ziegler

CS492A in Fall 2024

§3 Analog Computing

- Derivative+Integral Recap
- Recap: Classes of (Differential) *Equations*
- Engineering Bush's Differential Analyzer
- Shannon's Mathematical GPAC
  (*General Purpose Analog Computer*)
- Consequences and Properties of DAEs
  (*Differential Algebraic Equation Systems*)
- Analog Zeno-Effect: no (naïve) complexity

§3 Discrete vs. Continuous



$g:\mathbb{R}\to\mathbb{R}$ $\quad g:[0;1]\to\mathbb{R}$ $\quad f:\mathbb{N}\to\mathbb{N}$ $\quad f:\{\mathbf{0},\mathbf{1}\}^*\to\{\mathbf{0},\mathbf{1}\}^*$

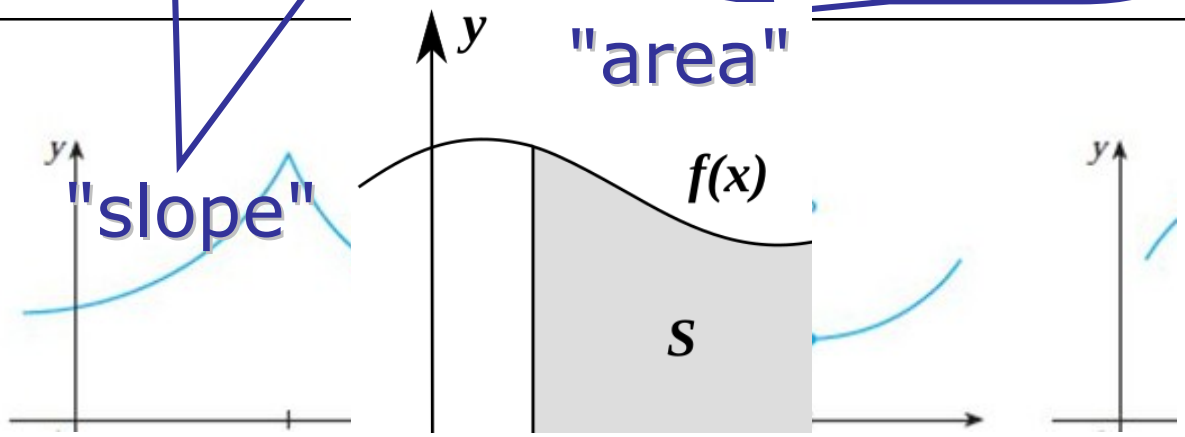
$g:\mathbb{R}^d\to\mathbb{R}$ $\quad g:[0;1]^d\to\mathbb{R}$ $\quad f:\mathbb{N}^d\to\mathbb{N}$

continuous? differentiable?

Theorem (Brouwer):
No non-empty open $U\subseteq\mathbb{R}^d$ is *homeomorphic* to any open $V\subseteq\mathbb{R}^e$, unless $d=e$.

Bijective pairing function ("*Hilbert Hotel*") $\langle x,y\rangle :=$
$:= x + (x+y)\cdot(x+y+1)/2$

§3 Derivative + Integral Recap



$$\mathrm{d}/_{\mathrm{d}x}\, x^n = n\cdot x^{n-1}$$

$$\ln'(t) = 1/t$$

$$\exp' = \exp,$$

$$\sin' = \cos,$$

$$\cos' = -\sin$$

$$(f\cdot g)' = f'\cdot g + f\cdot g'$$

$$\mathrm{d}/_{\mathrm{d}x}\, f(g(x)) = f'(g(x))\cdot g'(x)$$

$g{:}\subseteq\mathbb{R}^1\to\mathbb{R}$ $\quad g'(x)\,,\ \mathrm{d}/_{\mathrm{d}y}\, g(y)\,,\ \dot{g}(t)$

$g{:}\subseteq\mathbb{R}^d\to\mathbb{R}$ $\quad \partial_y g(x,y,z) \quad \partial_2 g(x_1,..,x_d)$

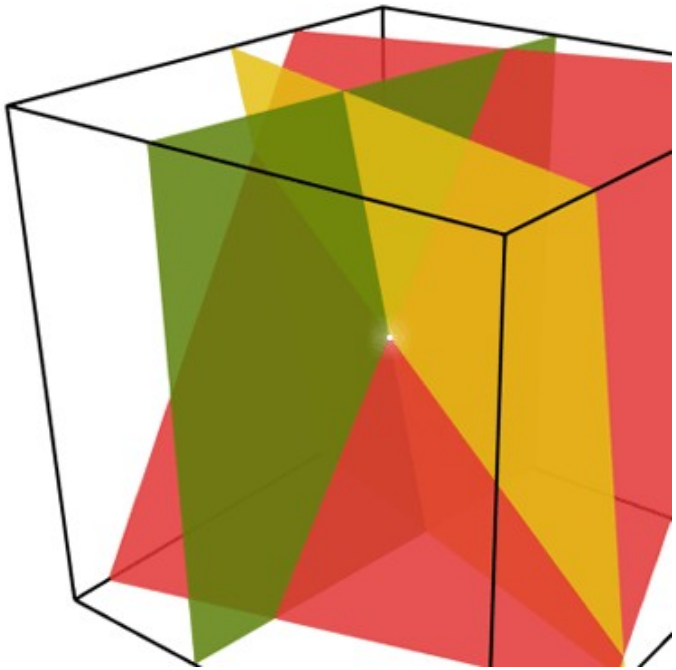
Stieltjes Integral $\quad \int_a^b f(t)\,\mathrm{d}g(t)$

$$= \int_a^b f(t)\cdot g'(t)\,\mathrm{d}t \quad \text{when } g' \text{ continuous}$$

$\int_a^b f(t)\,\mathrm{d}t$ $\qquad F(x) = \int_a^x f(t)\,\mathrm{d}t$ *anti*-derivative

("Fundamental Theorem of Calculus")

§3 *(Function)* Equations



a) $x^2 + 4x + 4 = 0$

b) $y^2 - 2 = 0 \quad (y>0)$

c) $y^2 + 1 = 0$

d)
$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 4 \end{aligned}$$

e) $e(t+s) = e(t) \cdot e(s)\,, \quad e(0) = 1$

Questions: i) Do/es the equation/s *have* a solution?

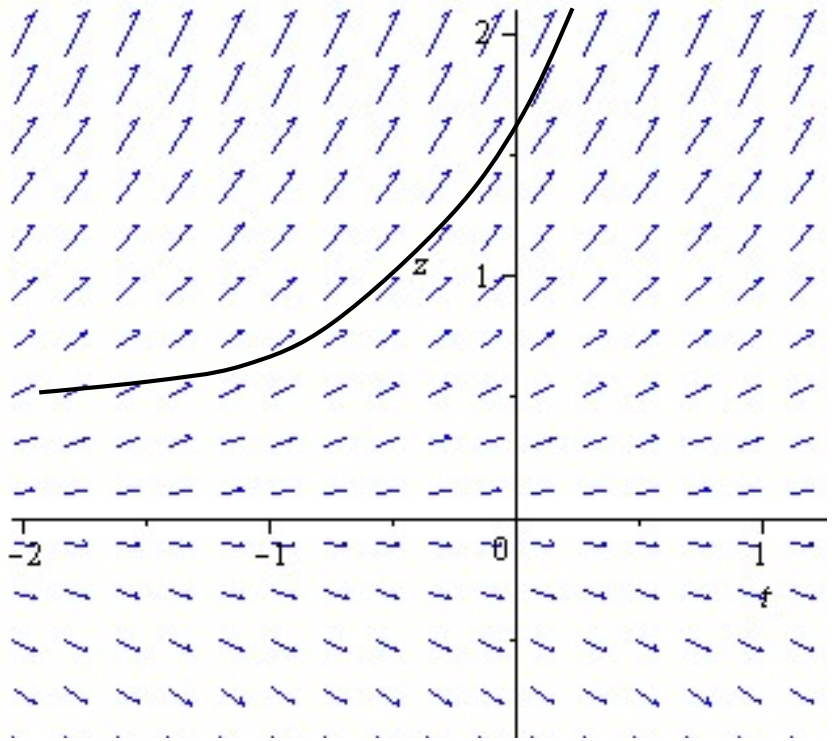
ii) In which mathematical "space" ?

iii) Is the solution unique?

$e:(0;\infty) \to \mathbb{R}$
continuous

§3 *Differential* Equations

f) $f'(t) = f(t)$ $\qquad f(0) = 1$



2D vector field $v(t,y)=(1,y)$

e) $e(t+s) = e(t) \cdot e(s)$, $\quad e(0) = 1$

Questions: i) Do/es the equation/s *have* a solution?

ii) In which mathematical "space" ?

iii) Is the solution unique?

$f : \mathbb{R} \to \mathbb{R}$

differentiable

§3 *Differential* Equations, Examples

f) $f'(t) = f(t)$ $\qquad f(0) = 1$

g) $g'(t) = t^k$ $\qquad (t>0)\,,\ g(1) = 0$ $\qquad k=3,2,1,0,-1,-2,-3,..$

h) $h'(t) = y(t)$ $\qquad (t>0)\,,\ h(0) = 0$

u) $u''(t) = -u(t)\ ,\ \ u(0) = 0\,,\ u'(0) = 1$

v) $u'(t)=v(t)\,,\ v'(t)=-u(t),\ u(0)=0,\ v(0)=1$

w) $w'(t) = \sqrt{w(t)}\ ,\ \ w(0)=0$

$$\boxed{\begin{aligned} w(t) &:= 0 && \text{for } t\leq C, \\ &:= (t-C)^2/4 && \text{for } t\geq C \end{aligned}}$$

Questions: i) Do/es the equation/s *have* a solution?

ii) In which mathematical "space" ? $\quad g,h,u,v,w : (0;\infty)\to\mathbb{R}$

iii) Is the solution unique? $\quad$ *continuously differentiable*

§3 *Differential* Equations, Classified

f) $f'(t) = f(t)$ $\quad f(0) = 1$ $\quad$ 1st order, linear, homogen.

g) $g'(t) = t^k$ $\quad (t>0)\,,\ g(1) = 0$ $\quad k=3,2,1,0,-1,-2,-3,..$

h) $h'(t) = y(t)$ $\quad (t>0)\,,\ h(0) = 0$ $\quad$ 1st ord. lin. *in*homog

u) $u''(t) = -u(t)\ ,\ u(0) = 0\,,\ u'(0) = 1$ $\quad$ **2nd** ord.lin.homog.

v) $u'(t)=v(t)\,,\ v'(t)=-u(t),\ u(0)=0,\ v(0)=1$ $\quad$ 1st ord. *system*

w) $w'(t) = \sqrt{w(t)}\ ,\ w(0)=0$ $\quad$ 1st order, *non*-lin., homog.

x) $x'(t) = y(t)\cdot x(t)\,,\ x(0)=1$ $\quad$ linear *non*-const. coefficient

y) $y'(t)=3,\ y(0)=0$ $\quad$ x,y system, *non*-lin., *autonomous*

z) $z'^2(t) - z(t) = 0,\ z(0)=0$ $\quad$ 1st order non-lin., *implicit*

explicit: $z'(t) = \ldots$

§3 *Differential* Equations, QUIZ

f) $f'(t) = f(t)$ $\quad f(0) = 1$

g) $g'(t) = t^k$ $\quad (t>0)\,,\ g(1) = 0$ $\quad k=3,2,1,0,-1,-2,-3,..$

h) $h'(t) = y(t)$ $\quad (t>0)\,,\ h(0) = 0$

u) $u''(t) = -u(t)\ ,\ \ u(0) = 0\,,\ u'(0) = 1$

v) $u'(t)=v(t)\,,\ v'(t)=-u(t),\ u(0)=0,\ v(0)=1$

w) $w'(t) = \sqrt{w(t)}\ ,\quad w(0)=0$

Verify that the following *are* indeed solutions:

$f(t) := \exp(t),\qquad g(t) := (t^{k+1}-1)/(k+1),\qquad h(t) := \int_0^t y(s)\,\mathrm{d}s,$

$u(t) := A\cdot\sin(t)+B\cdot\cos(t),$

$v(t) := A\cdot\cos(t)-B\cdot\sin(t)$

$$w(t) := \begin{cases} 0 & \text{for } t\le C, \\ (t-C)^2/4 & \text{for } t\ge C \end{cases}$$

§3 Differential *Algebraic* Equations

Un $u_0(t) :\equiv 1$
$u_1(t) :\equiv t$

(*) $u_k'(t) = \sum_{i,j=0..n}^{j\neq k} c_{ijk} \cdot u_i(t) \cdot u_j'(t) \qquad k=2,\ldots,n \qquad c_{ijk} \in \mathbb{Q}$

*ex*plicit 1st order system polyn. *in*homog. *autonomous*

***Non*-example:** $z'(t) = \exp(z(t))$

Lemma: (*) can be rewritten as *one im*plicit *polynomial* differential equation with *non*-constant coefficients

(**) $0 \equiv \sum_k c_k \cdot t^{m_k} \cdot (v(t))^{m_{0,k}} \cdot (v'(t))^{m_{1,k}} \cdot (v''(t))^{m_{2,k}} \ldots (v^{(d)}(t))^{m_{d,k}}$

x) $x'(t) = y(t) \cdot x(t)$, $x(0)=1$ linear *non*-const. coefficient

y) $y'(t)=3$, $y(0)=0$ x,y system, *non*-lin., *autonomous*

Fact [Hölder/Hilbert]: (Analytic continun.) of Riemann ζ function and Γ function satisfy *no* equation of form (**).

§3 Bush's Differential Analyzer

1890 ~ 1974

rods turning back/forth

at angles $u(t)$, $v(t)$, ...

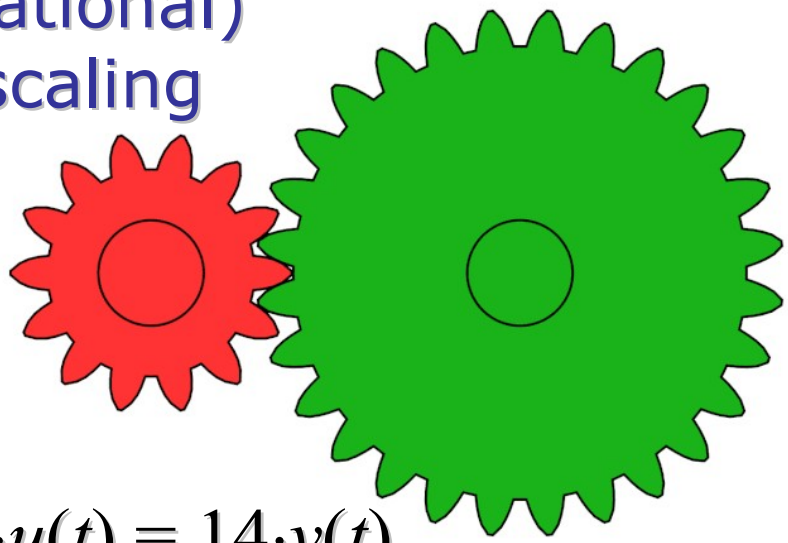
rods connected,

one driving:

$u_1(t) \equiv t$

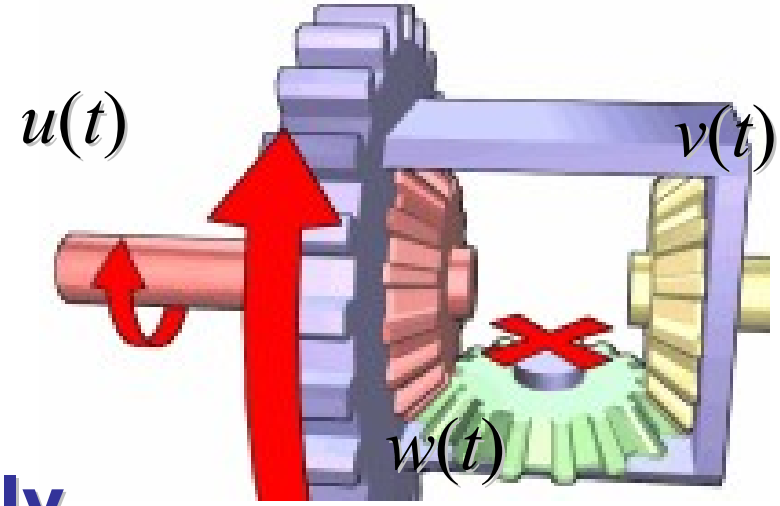
§3 Three Analog Connectives

(rational) scaling



$$28 \cdot u(t) = 14 \cdot v(t)$$

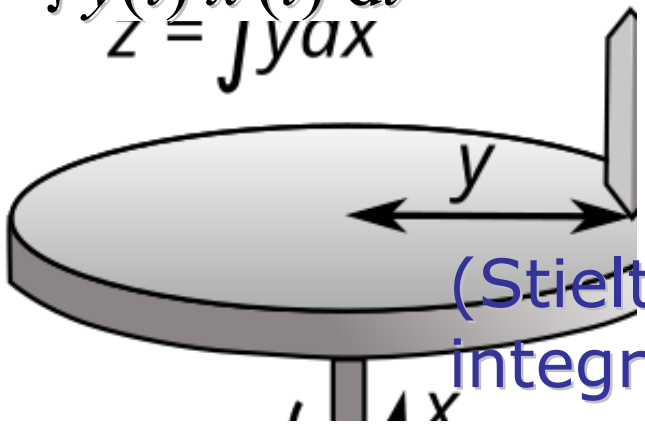
$$w(t) = (u(t) + v(t))/2$$



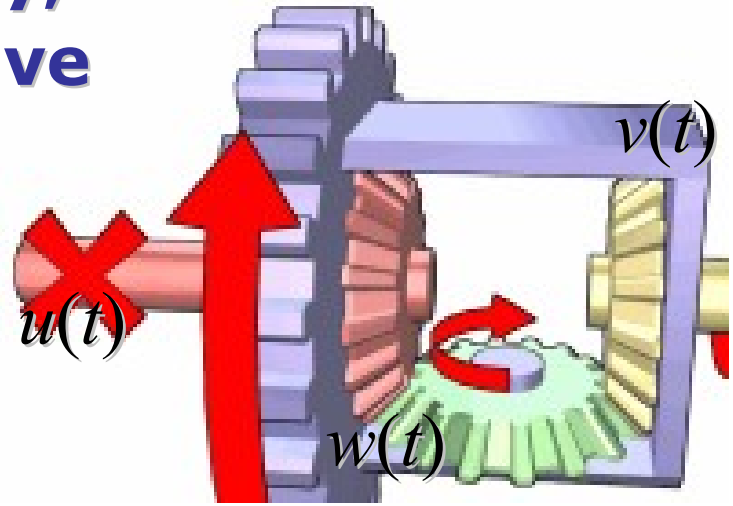
no multiply, no derivative

$$z(t) = \int y(t)\, \mathrm{d}x(t)$$
$$= \int y(t) \cdot x'(t)\, \mathrm{d}t$$

$$z = \int y dx$$



(Stieltjes) integration

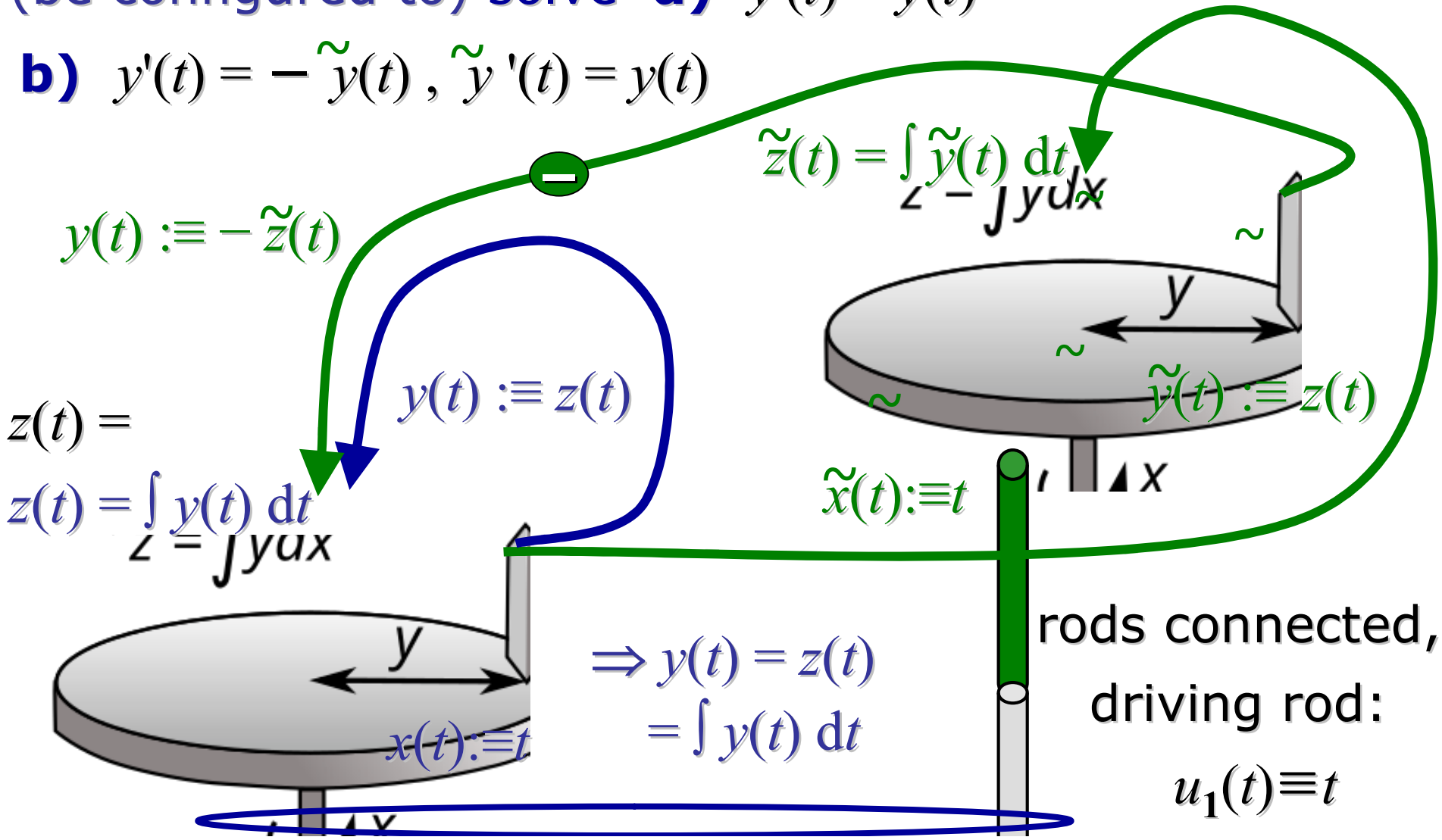


(binary) averaging

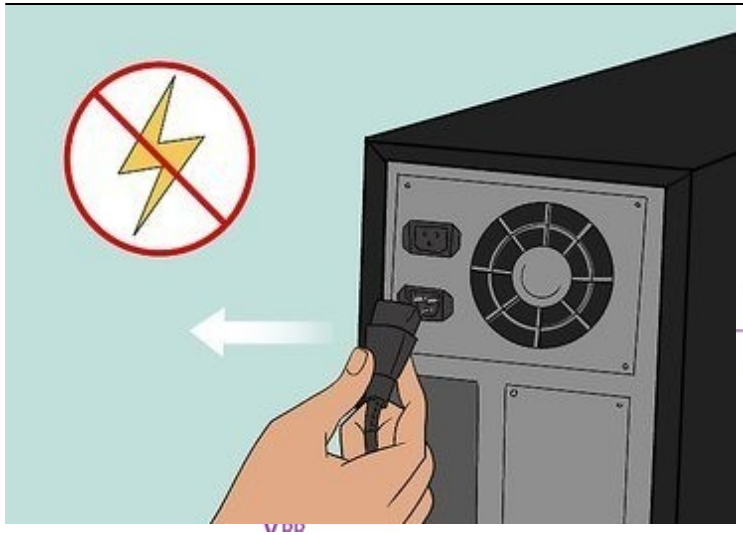
§3 Example Configurations

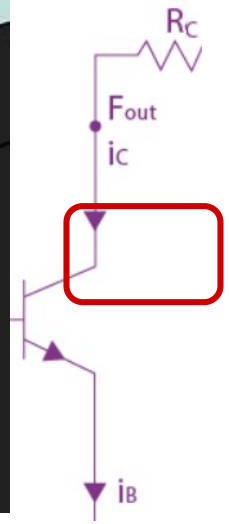
Examples: The Differential Analyzer can (be configured to) solve **a)** $y'(t) = y(t)$

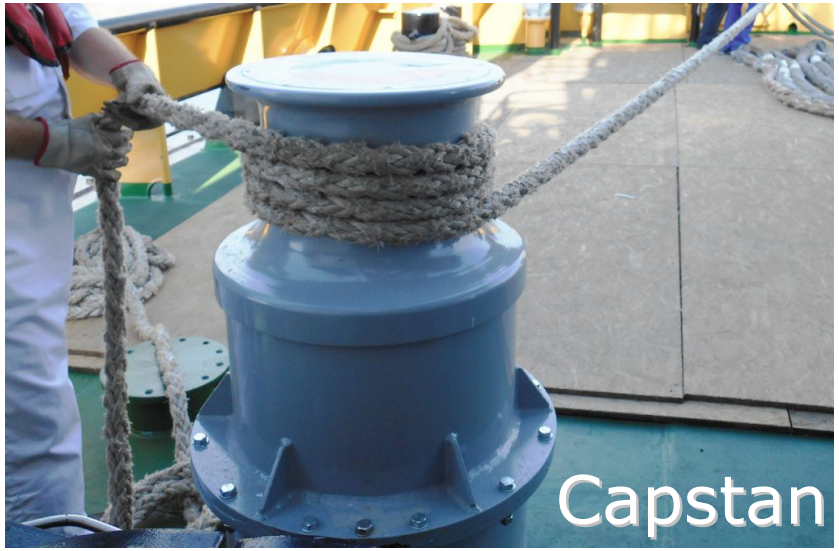
b) $y'(t) = -\tilde{y}(t)\,,\ \tilde{y}'(t) = y(t)$



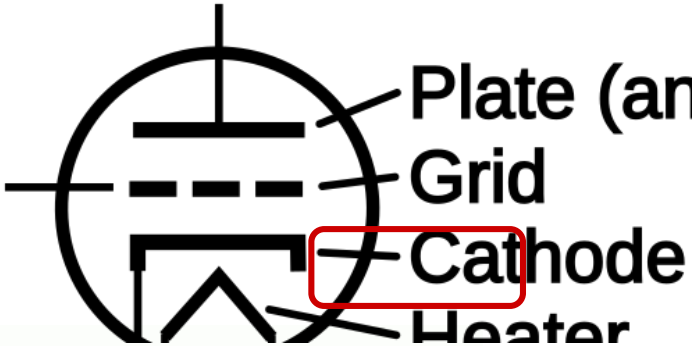
§3 *Torque* Amplifier

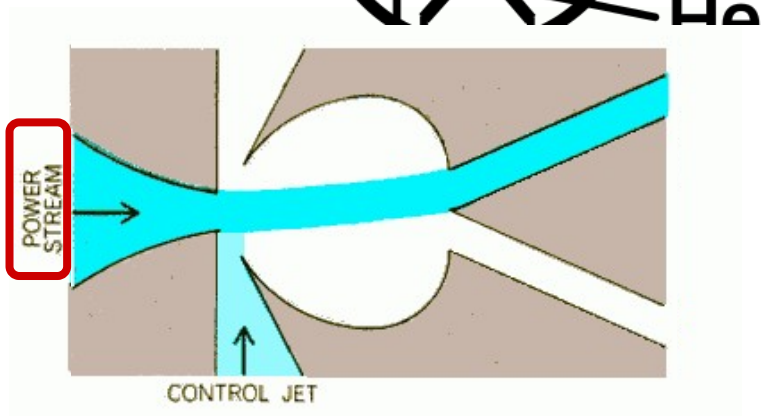


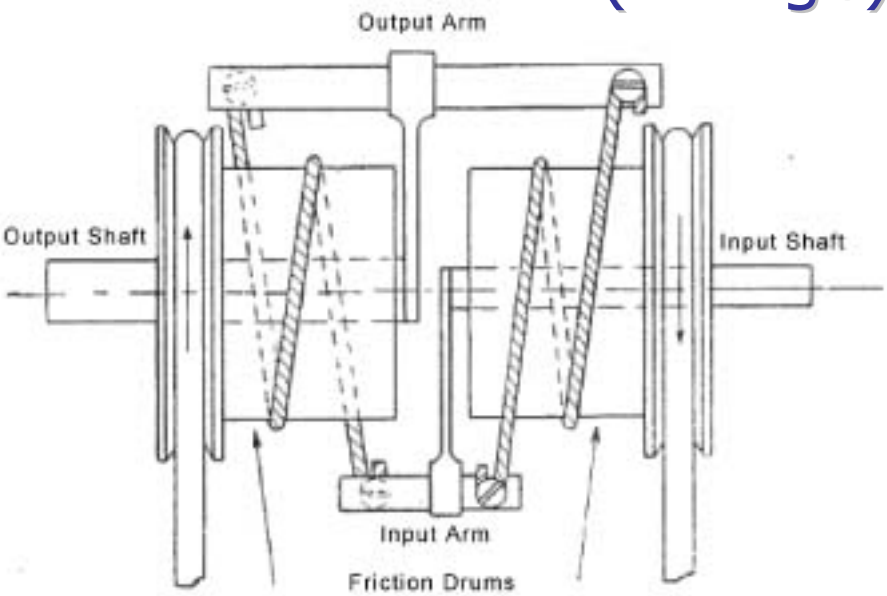




(≠winge)







§3 Another Example Config.

Examples: The Differential Analyzer can (be configured to) solve **a)** $y'(t) = y(t)$

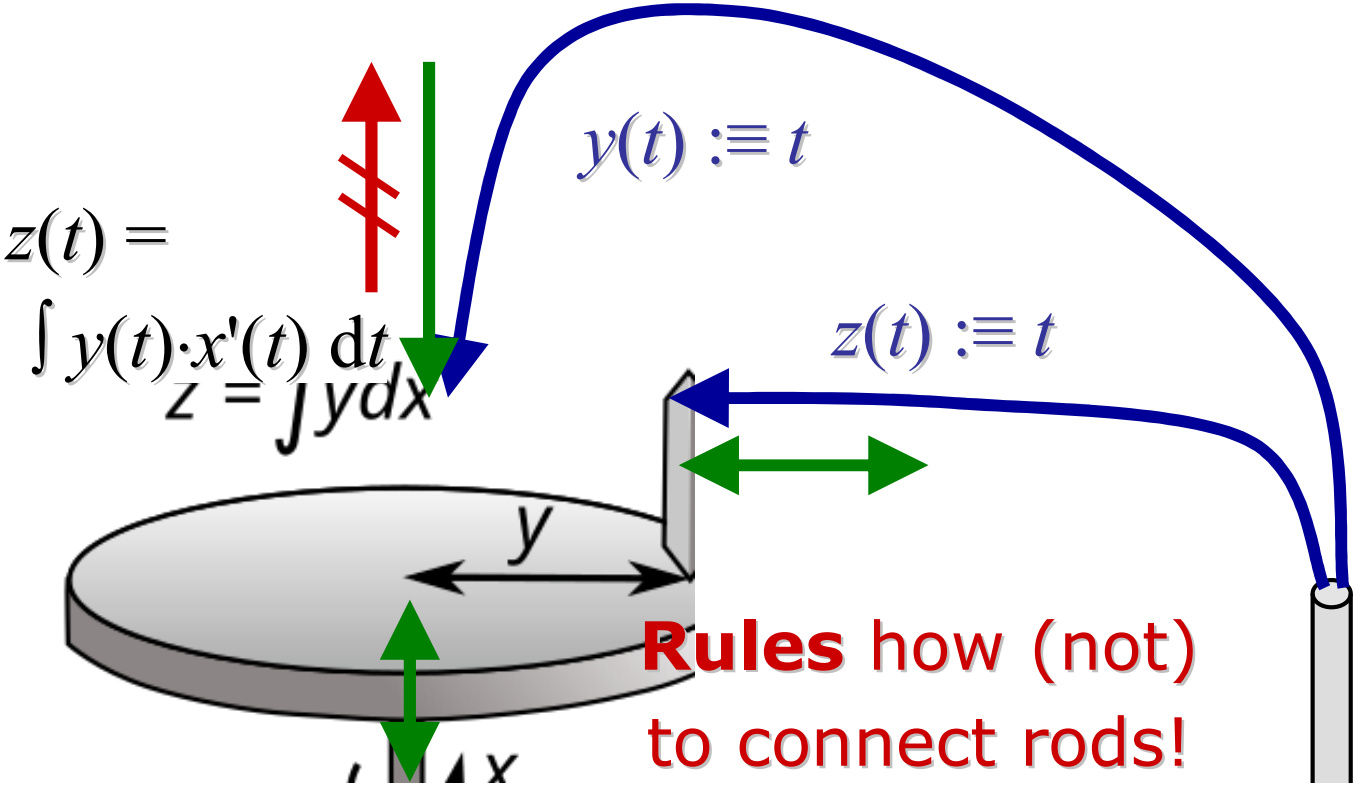
b) $y'(t) = -\tilde{y}(t)\,,\ \tilde{y}'(t) = y(t)$

c) $x'(t) = 1/t$

no multiplication, no differentiation

$t \equiv z(t)$
$= \int t \cdot x'(t)\,\mathrm{d}t$

$1 \equiv t \cdot x'(t)$



Rules how (not) to connect rods!

driving rod:
$u_1(t) \equiv t$

§3 Questions

Examples: The Differential Analyzer can (be configured to) **calculate** **a)** $\exp$

b) $\sin$ and $\cos$

c) $\ln$

no multiplication, no differentiation

d) $u_1(t) \equiv t$

e) Its calculations are closed under composition!

a) Mathematical model of the Differential Analyzer?

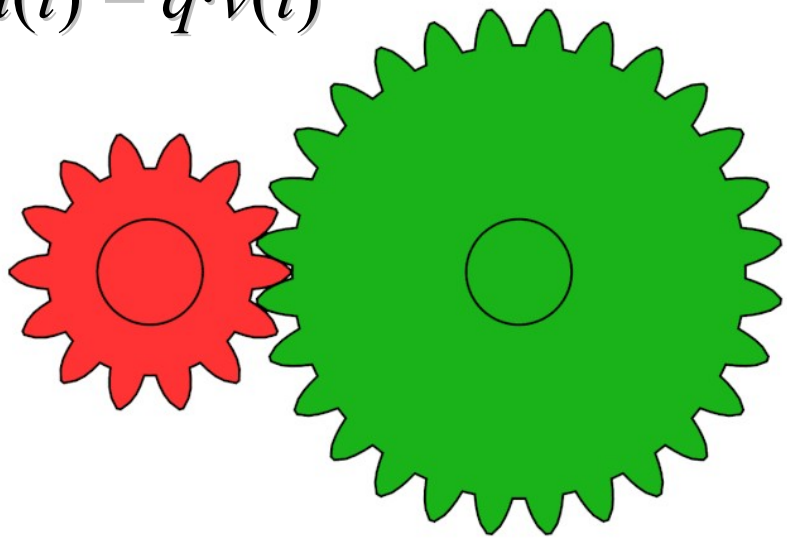
b/c) Which functions can/not be calculated by it?

Can it calculate multiplication/differentiation *indirectly*?

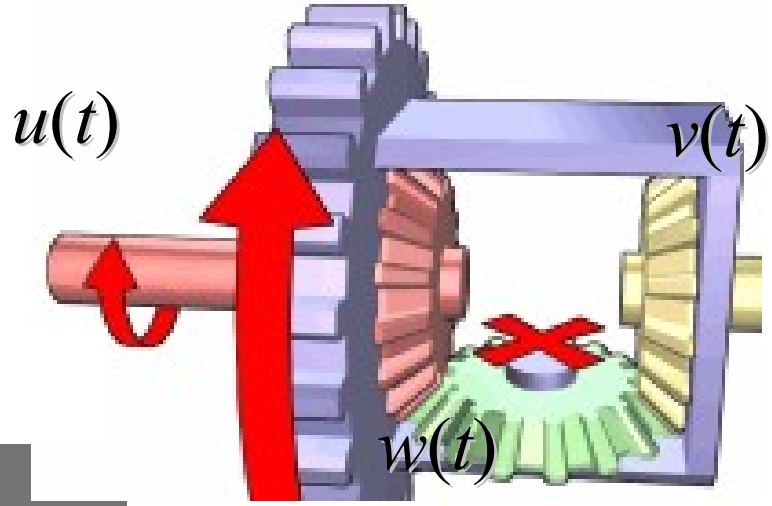
d) How configure/"program" to calculate a given function?

§3 Math of Analog Computing

$u(t) = q \cdot v(t)$

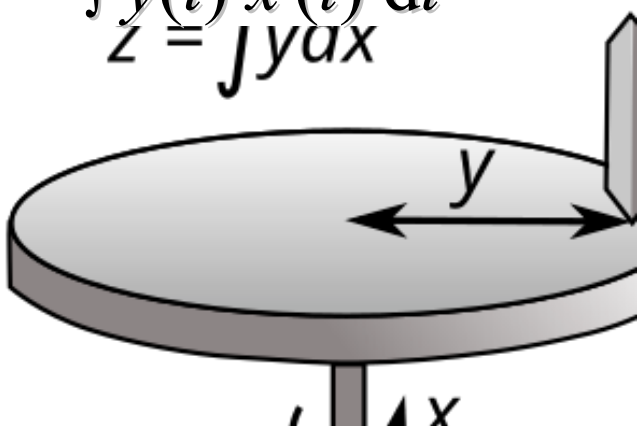


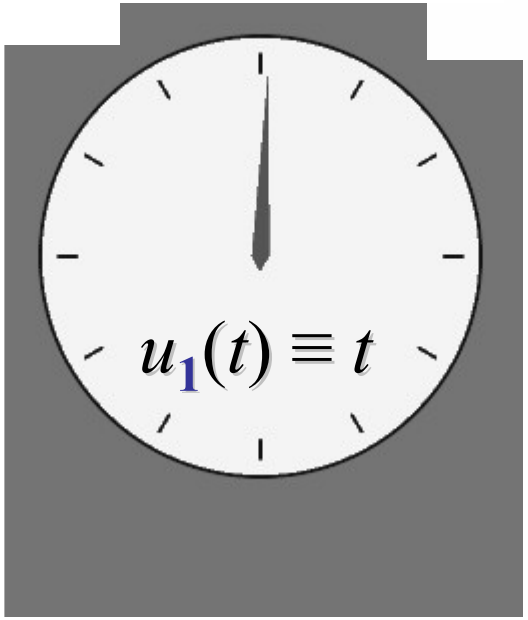
$w(t) = (u(t) + v(t))/2$



$z(t) = \int y(t)\, \mathrm{d}x(t)$

$= \int y(t) \cdot x'(t)\, \mathrm{d}t$





$u_0(t) \equiv 1$

§3 Shannon: *GPAC* = *DAEs*

Definition: Let ***GPAC*** denote the least class of functions $u=u(t)$ containing u_0 and u_1 and closed under **(iii)~(v).**

i) $u_0(t) :\equiv 1$

ii) $u_1(t) :\equiv t$

iii) $v(t) \to u(t) = \boxed{q} \cdot v(t)$ $\quad q \in \mathbb{Q}$

iv) $u(t), v(t) \to w(t) = (u(t) + v(t))/2$

v) $x(t), y(t) \to z(t) = \int y(t) \cdot x'(t)\, \mathrm{d}t$

no multiplication, no differentiation

Theorem: a) If $u_2(t),\ldots,u_n(t)$ are generated by [some configuration of] the Differential Analyzer, then it holds

(*) $u_k'(t) = \sum_{i,j=0..n}^{j\neq k} c_{ijk} \cdot u_i(t) \cdot u_j'(t) \qquad k=2,\ldots,n \qquad c_{ijk} \in \mathbb{Q}$

b) If $u_2(t),\ldots,u_n(t)$ satisfy (*),
then they can be generated by the Differential Analyzer.

§3 Proof: GPAC = DAEs

i) $u_0(t) :\equiv 1$

ii) $u_1(t) :\equiv t$

iii) $v(t) \to u(t) = \boxed{q} \cdot v(t)$ $\quad q \in \mathbb{Q}$

iv) $u(t), v(t) \to w(t) = (u(t) + v(t))/2$

v) $x(t), y(t) \to z(t) = \int y(t) \cdot x'(t)\, \mathrm{d}t$

First consider only "*linear*" connectives (i)~(iv):

Lemma a'): Rods driven/connected by (i)~(iv) satisfy

(***) $u_k(t) = \sum_{j=0..n}^{j \neq k} a_{jk} \cdot u_j(t)$ $\quad a_{jk} \in \mathbb{Q}$

b') Any system of linear conditions (**) among a [sub-] collection of rods can be enforced using linear (i)~(iv).

Proof: by induction.

§3 Proof: *GPAC* $\subseteq$ *DAEs*

Now consider only *integrator* connectives (v):

v) $x(t),y(t) \to z(t) = \int y(t)\cdot x'(t)\,\mathrm{d}t$

Theorem: a) If $u_2(t),\ldots,u_n(t)$ are generated by [some configuration of] the Differential Analyzer, then it holds

(*) $u_k'(t) = \sum_{i,j=0..n}^{j\neq k} c_{ijk}\cdot u_i(t)\cdot u_j'(t)$ $\quad k=2,\ldots,n \quad c_{ijk}\in\mathbb{Q}$

(***) $u_k(t) = \sum_{j=0..n}^{j\neq k} a_{jk}\cdot u_j(t)$ $\quad a_{jk}\in\mathbb{Q}$

Proof: By (***), $\int$−inputs $x(t),y(t)$ have the form

$$x(t) = \sum_{j=0..n} a_{jk}\cdot u_j(t) \qquad y(t) = \sum_{i=0..n} b_{ik}\cdot u_i(t)$$

$\Rightarrow\ z(t) = \int y(t)\cdot x'(t)\,\mathrm{d}t$ has $z'(t) = y(t)\cdot x'(t) = \ldots$

§3 Proof: *GPAC* $\supseteq$ *DAEs*

Now consider only *integrator* connectives (v):

v) $x(t),y(t) \to z(t) = \int y(t)\cdot x'(t)\,\mathrm{d}t$

Theorem: b) If $u_2(t),\ldots,u_n(t)$ satisfy (*),
then they can be generated by the Differential Analyzer.

(*) $u_k'(t) = \sum_{i,j=0..n}^{j\neq k} c_{ijk}\cdot u_i(t)\cdot u_j'(t)$ $\quad k=2,\ldots,n \quad c_{ijk}\in\mathbb{Q}$

(***) $u_k(t) = \sum_{j=0..n}^{j\neq k} a_{jk}\cdot u_j(t)$ $\quad a_{jk}\in\mathbb{Q}$

$$(*) \Rightarrow u_k(t) = \int u_k'(t)\,\mathrm{d}t = \sum_{i,j=0..n}^{j\neq n} c_{ijn}\cdot\int u_i(t)\cdot u_j'(t)\,\mathrm{d}t$$

combine these u_{ij} according to (***)

make outputs u_{ij} of *integrators*

§3 Consequences *GPAC* = *DAEs*

Corollary: If $x(t),y(t)$ can be generated,

then so can $x'(t)$ and $x(t)\cdot y(t)$.

no multiplication, no differentiation

v) $x(t),y(t) \to z(t) = \int y(t)\cdot x'(t)\,\mathrm{d}t$

Question: HOW??

Summary: $u_2(t),\ldots,u_n(t)$ can be generated by [some configuration of] the Differential Analyzer **iff** it holds

(*) $u_k'(t) = \sum_{i,j=0..n}^{j\neq k} c_{ijk}\cdot u_i(t)\cdot u_j'(t)$ $\quad k=2,\ldots,n \quad c_{ijk}\in\mathbb{Q}$

Corollary: $v(t)$ can be generated iff satisfies an implicit polynomial differential equation with non-const. coeff.s

(**) $0 \equiv \sum_k c_k\cdot t^{m_k}\cdot (v(t))^{m_{0,k}}\cdot (v'(t))^{m_{1,k}}\cdot (v''(t))^{m_{2,k}}\cdots (v^{(d)}(t))^{m_{d,k}}$

Corollary [Hölder/Hilbert]: ζ and Γ can*not* be generated

§3 Analog Zeno Effect

Consider differential equation $y'(t) = f(y(t)) \quad y(1) = y_0$

for fixed continuous $f:\mathbb{R}\to\mathbb{R}$

with solution $y(t),\ y:\mathbb{R}\to\mathbb{R}$.

(1D autonom. 1st-order)

Now consider <u>modified</u> differential equation *system*

$$z'(t) = f(z(t)) \cdot w(t) \qquad w'(t) = w(t) \qquad z(0) = y_0 \qquad w(0) = 1$$

with solution $w(t) = \exp(t)$, $\boxed{z(t) = y(\exp(t))}$

Quiz a): Verify this solution!

b) *Doubly* exponential accelerating differential equation?

Differential equations can be "*accelerated*"!

see [doi:10.1016/j.jco.2016.05.002]

no (naïve definition of) time-complexity…

§3 Analog Computing

- Derivative+Integral Recap
- Recap: Classes of (Differential) *Equations*
- Engineering Bush's Differential Analyzer
- Shannon's Mathematical GPAC
  (*General Purpose Analog Computer*)
- Consequences and Properties of DAEs
  (*Differential Algebraic Equation Systems*)
- Analog Zeno-Effect: no (naïve) complexity