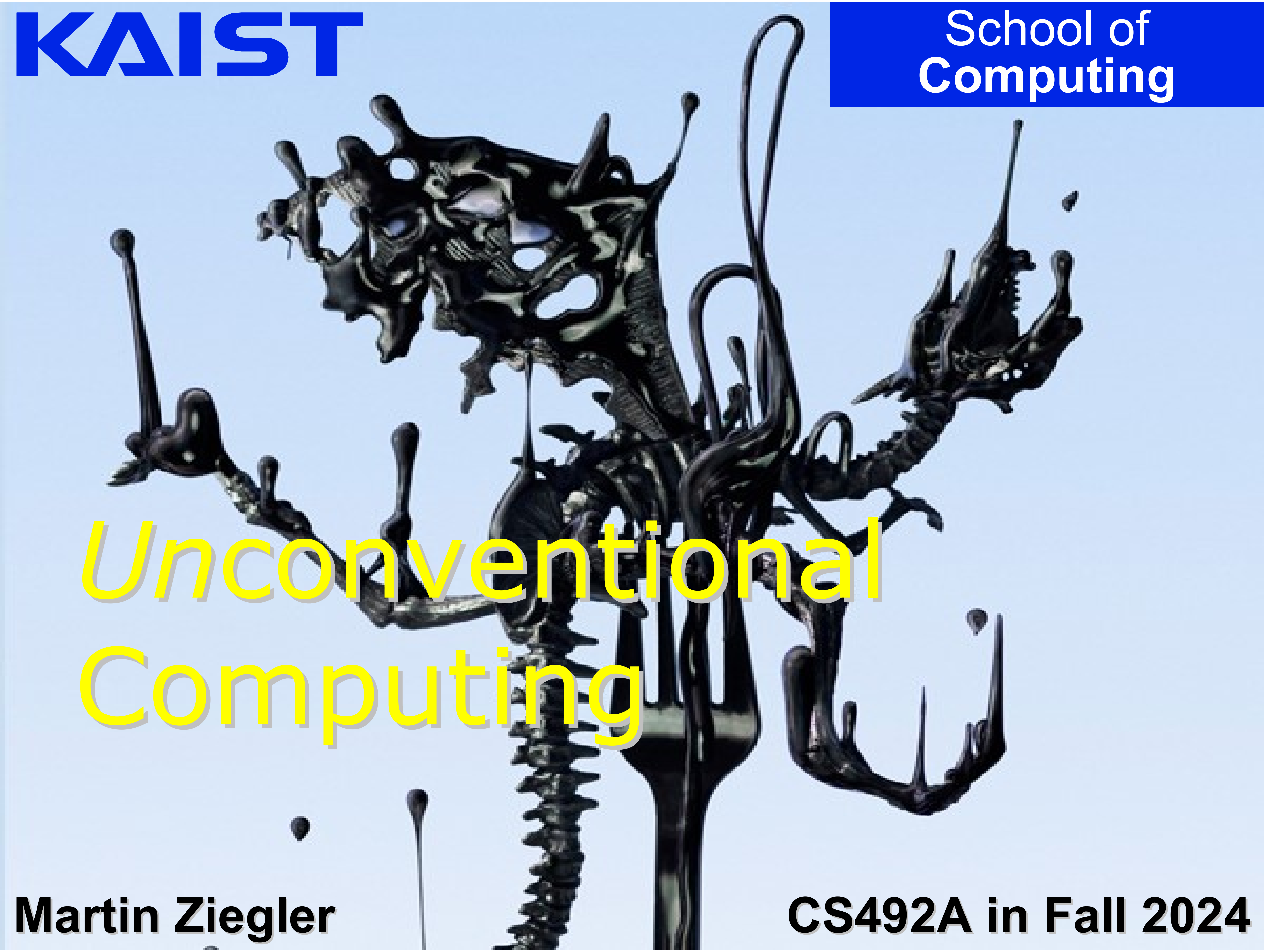
KAIST

School of Computing



*Un*conventional Computing

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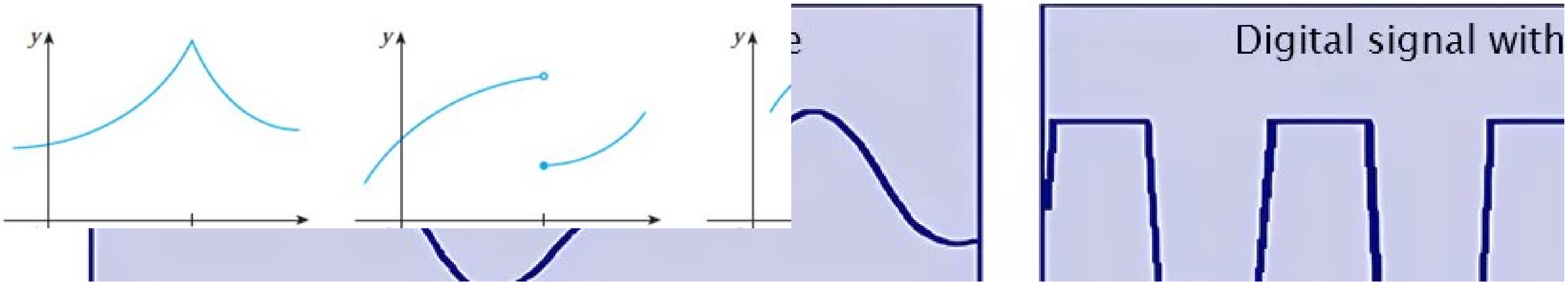
CS492A in Fall 2024

§3 Analog Computing

- Derivative+Integral Recap
- (Differential) *Equations*
- Bush's Differential Analyzer
- Shannon's Mathematical Theory
- Zeno-Effect: no complexity

§3 Discrete vs. Continuous



$g:\mathbb{R} \to \mathbb{R}$ $\quad g:[0;1] \to \mathbb{R}$ $\quad f:\mathbb{N} \to \mathbb{N}$

$g:\mathbb{R}^d \to \mathbb{R}$ $\quad g:[0;1]^d \to \mathbb{R}$ $\quad f:\mathbb{N}^d \to \mathbb{N}$

$f:\{\mathbf{0},\mathbf{1}\}^* \to \{\mathbf{0},\mathbf{1}\}^*$

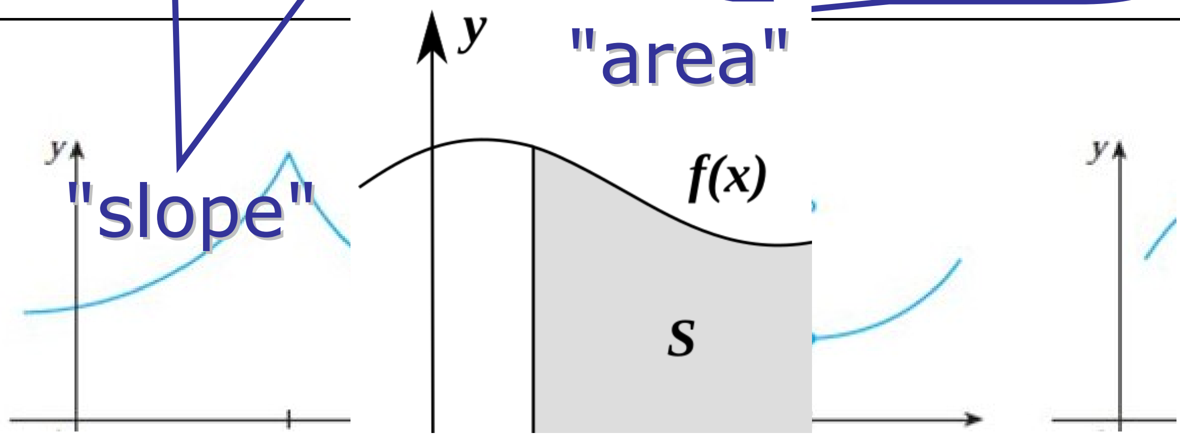
continuous? differentiable?

Theorem (Brouwer):
No non-empty open $U \subseteq \mathbb{R}^d$ is *homeomorphic* to any open $V \subseteq \mathbb{R}^e$, unless $d=e$.

Bijective pairing function ("*Hilbert Hotel*") $\langle x,y \rangle :=$

$:= x + (x+y)\cdot(x+y+1)/2$

§3 Derivative+Integral Recap



$g :\subseteq \mathbb{R}^1 \to \mathbb{R}$ $\quad g'(x)$, ${}^{\mathrm{d}}/_{\mathrm{d}y}\, g(y)$, $\dot{g}(t)$

$g :\subseteq \mathbb{R}^d \to \mathbb{R}$ $\quad \partial_y g(x,y,z) \quad \partial_2 g(x_1,\dots,x_d)$

Stieltjes Integral $\quad \int_a^b f(t)\, \mathrm{d}g(t)$

$= \int_a^b f(t) \cdot g'(t)\, \mathrm{d}t$ when g' continuous

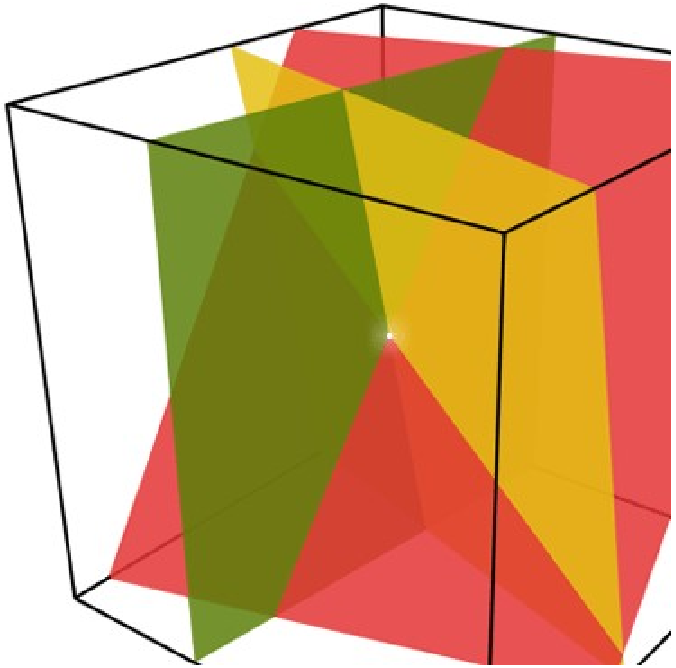
$\int_a^b f(t)\, \mathrm{d}t$

$F(x) = \int_a^x f(t)\, \mathrm{d}t$ *anti*-derivative

("Fundamental Theorem of Calculus")

$${}^{\mathrm{d}}/_{\mathrm{d}x}\, x^n = n \cdot x^{n-1}$$
$$\ln'(t) = 1/t$$
$$\exp' = \exp,$$
$$\sin' = \cos,$$
$$\cos' = -\sin$$
$$(f \cdot g)' = f' \cdot g + f \cdot g'$$
$${}^{\mathrm{d}}/_{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x)$$

§3 *(Function)* Equations



a) $x^2 + 4x + 4 = 0$

b) $y^2 - 2 = 0 \quad (y>0)$

c) $y^2 + 1 = 0$

d)
$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 4 \end{aligned}$$

e) $e(t+s) = e(t) \cdot e(s)\,, \quad e(0) = 1$

Questions: i) Do/es the equation/s *have* a solution?

ii) In which mathematical "space" ?

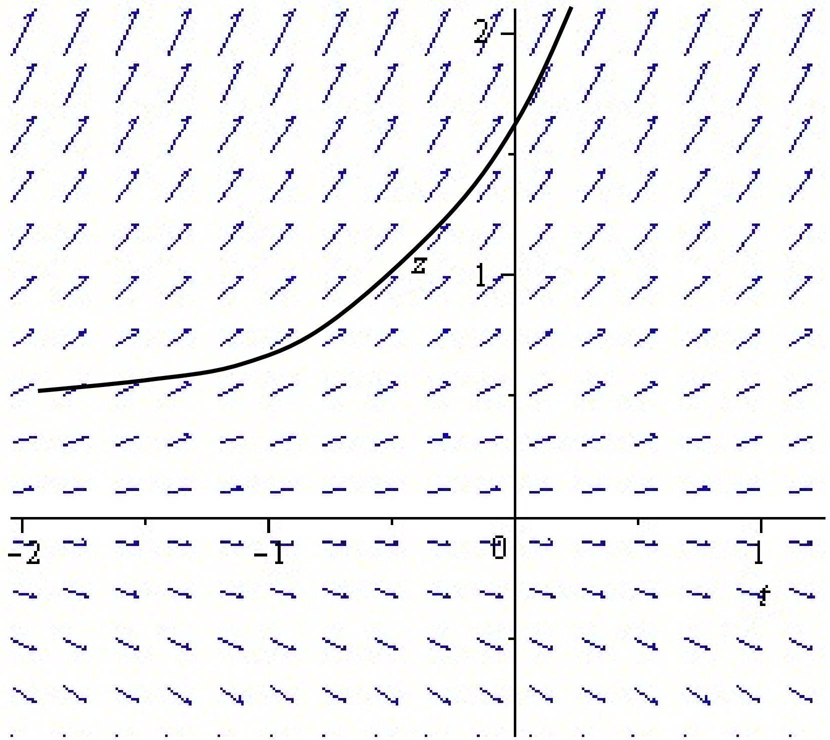
iii) Is the solution unique?

$e:(0;\infty) \to \mathbb{R}$

continuous

§3 *Differential* Equations

f) $f'(t) = f(t)$ $\qquad f(0) = 1$



2D vector field $v(t,y)=(1,y)$

e) $e(t+s) = e(t) \cdot e(s)\ , \quad e(0) = 1$

Questions: i) Do/es the equation/s *have* a solution?

ii) In which mathematical "space" ? $\qquad f: \mathbb{R} \to \mathbb{R}$ *differentiable*

iii) Is the solution unique?

§3 *Differential* Equations, Examples

f) $f'(t) = f(t)$ $\qquad f(0) = 1$

g) $g'(t) = t^k$ $\qquad (t>0)\,,\; g(1) = 0$ $\qquad k=3,2,1,0,-1,-2,-3,..$

h) $h'(t) = y(t)$ $\qquad (t>0)\,,\; h(0) = 0$

u) $u''(t) = -u(t)\;,\quad u(0) = 0\,,\; u'(0) = 1$

v) $u'(t)=v(t)\,,\; v'(t)=-u(t),\; u(0)=0,\; v(0)=1$

w) $w'(t) = \sqrt{w(t)}\;,\quad w(0)=0$

$$w(t) := 0 \quad \text{for } t\leq C,$$
$$:= (t-C)^2/4 \quad \text{for } t\geq C$$

Questions: i) Do/es the equation/s *have* a solution?

ii) In which mathematical "space" ? $\quad g,h,u,v,w : (0;\infty)\to\mathbb{R}$

iii) Is the solution unique? $\quad$ *continuously differentiable*

§3 *Differential* Equations, QUIZ

f) $f'(t) = f(t)$ $f(0) = 1$

g) $g'(t) = t^k$ $(t>0)$, $g(1) = 0$ $k=3,2,1,0,-1,-2,-3,..$

h) $h'(t) = y(t)$ $(t>0)$, $h(0) = 0$

u) $u''(t) = -u(t)$, $u(0) = 0$, $u'(0) = 1$

v) $u'(t)=v(t)$, $v'(t)=-u(t)$, $u(0)=0$, $v(0)=1$

w) $w'(t) = \sqrt{w(t)}$, $w(0)=0$

Verify that the following *are* indeed solutions:

$f(t) := \exp(t)$, $g(t) := (t^{k+1}-1)/(k+1)$, $h(t) := \int^t y(s)\,ds$,

$u(t) := A\cdot\sin(t)+B\cdot\cos(t)$,

$v(t) := A\cdot\cos(t)-B\cdot\sin(t)$

$w(t) := 0$ for $t\leq C$,

$:= (t-C)^2/4$ for $t\geq C$

§3 *Differential* Equations, Classified

f) $f'(t) = f(t)$ $\quad f(0) = 1$ — 1st order, linear, homogen.

g) $g'(t) = t^k$ $\quad (t>0)\,,\; g(1) = 0$ — $k=3,2,1,0,-1,-2,-3,..$

h) $h'(t) = y(t)$ $\quad (t>0)\,,\; h(0) = 0$ — 1st ord. lin. *in*homog

u) $u''(t) = -u(t)\,,\quad u(0) = 0\,,\; u'(0) = 1$ — **2**nd ord.lin.homog.

v) $u'(t)=v(t)\,,\; v'(t)=-u(t),\; u(0)=0,\; v(0)=1$ — 1st ord. *system*

w) $w'(t) = \sqrt{w(t)}\,,\quad w(0)=0$ — 1st order, *non*-lin., homog.

x) $x'(t) = y(t)\cdot x(t)\,,\quad x(0)=1$ — linear *non*-const. coefficient

y) $y'(t)=2,\; y(0)=0$ — x,y system, *non*-lin., *autonomous*

z) $z'^2(t) - z(t) = 0,\; z(0)=0$ — 1st order non-lin., *implicit*

ex*plicit*: $z'(t) = \ldots$

§3 *Algebraic* Differential Equations

(*) $u_k'(t) = \sum_{i,j=\mathbf{0}..n} c_{ijk} \cdot u_i(t) \cdot u_j'(t)$ for $u_0(t) :\equiv t$, $c_{ijk} \in \mathbb{R}$

1st order system **polynom.** *in*homog. with *const*. coeff.s

***Non*-example:** $z'(t) = \exp(z(t))$

u) $u''(t) = -u(t)$, $u(0) = 0$, $u'(0) = 1$ **2**nd ord.lin.homog.

v) $u'(t) = v(t)$, $v'(t) = -u(t)$, $u(0)=0$, $v(0)=1$ 1st ord. *system*

w) $w'(t) = \sqrt{w(t)}$, $w(0)=0$ 1st order, *non*-lin., homog.

x) $x'(t) = y(t) \cdot x(t)$, $x(0)=1$ linear *non*-const. coefficient

y) $y'(t)=2$, $y(0)=0$ x,y system, *non*-lin., *autonomous*

Fact (Hölder/Hilbert): Analytic continuations of Riemann ζ function and Γ function satisfy *no* equation of form (*).

§3 Bush's Differential Analyzer

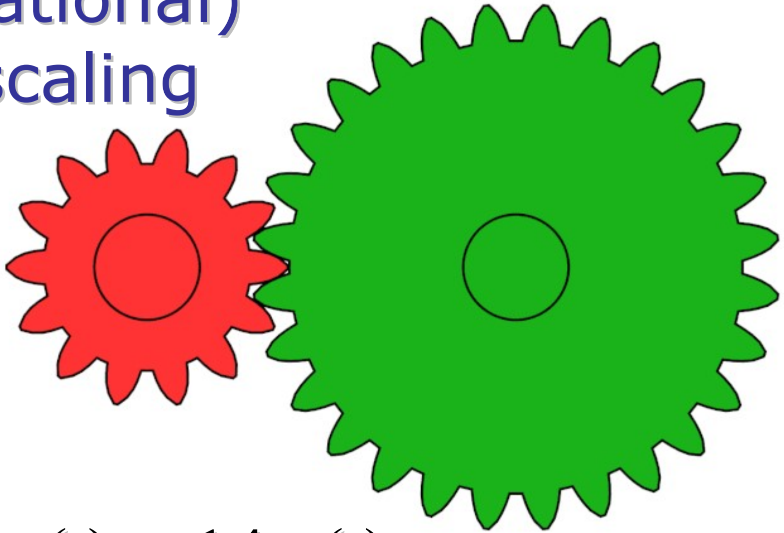
1890 ~ 1974

rods turning back/forth
at angles $u(t)$, $v(t)$, ...

rods connected,
one driving:
$u_0(t)\equiv t$

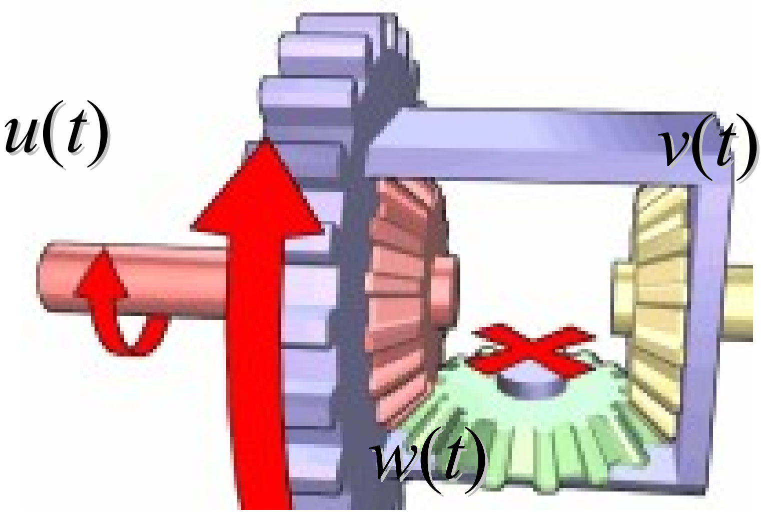
§3 Three Analog Primitives

(rational) scaling

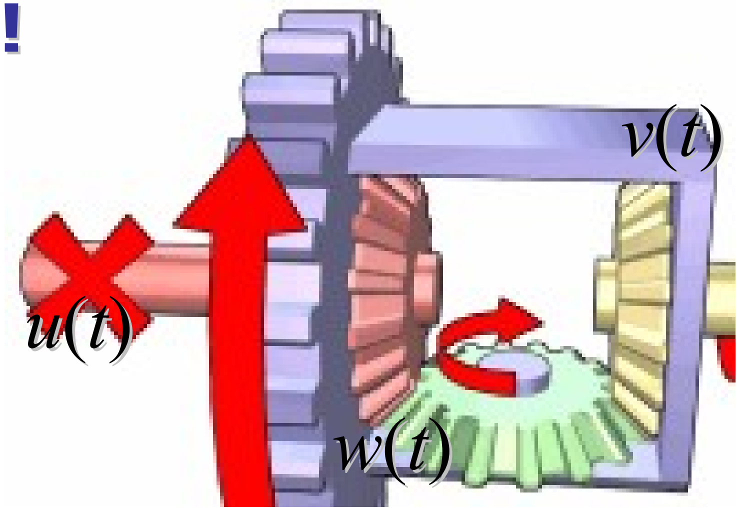


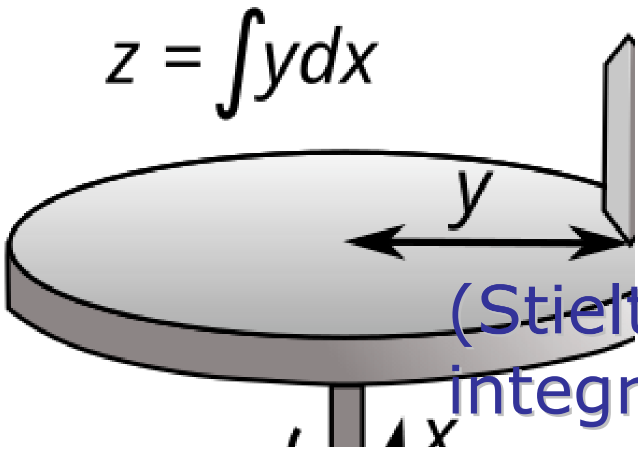
$28 \cdot u(t) = 14 \cdot v(t)$

$w(t) = (u(t) + v(t))/2$



no multiply!

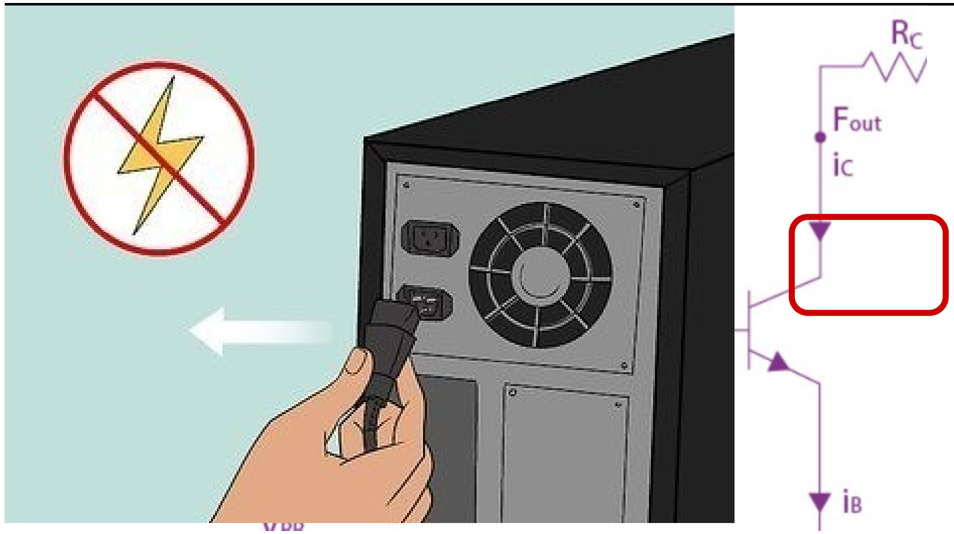


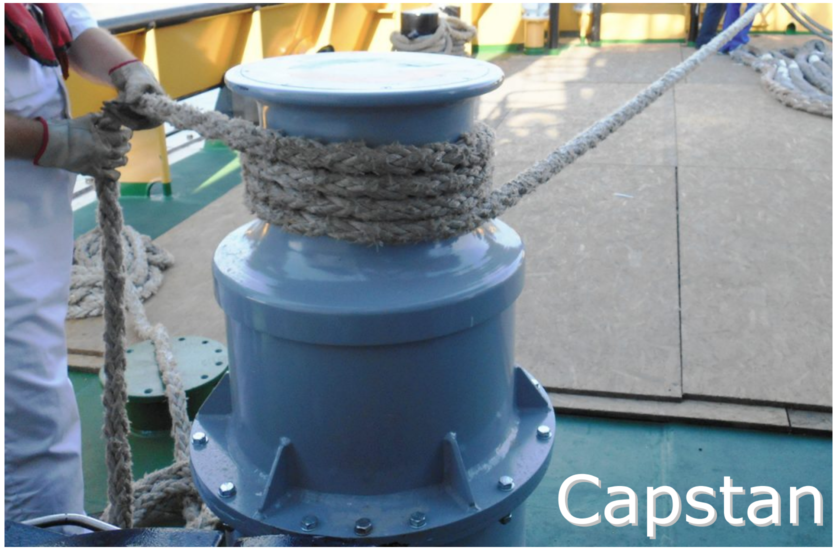


(Stieltjes) integration

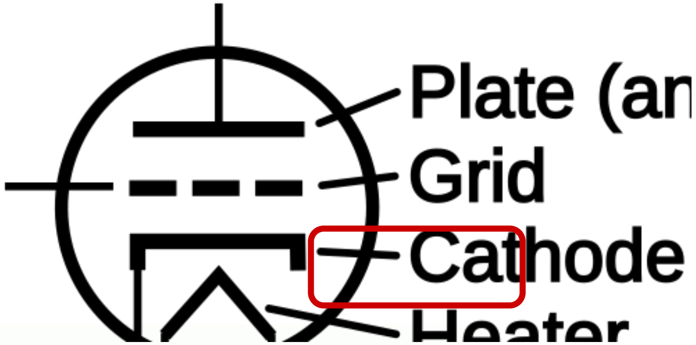
(binary) averaging

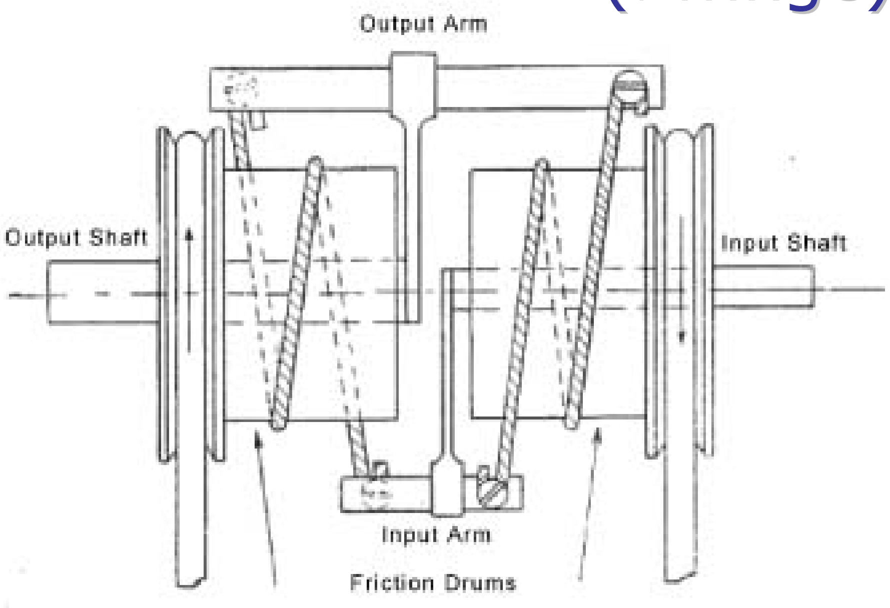
§3 ***Torque*** Amplifier

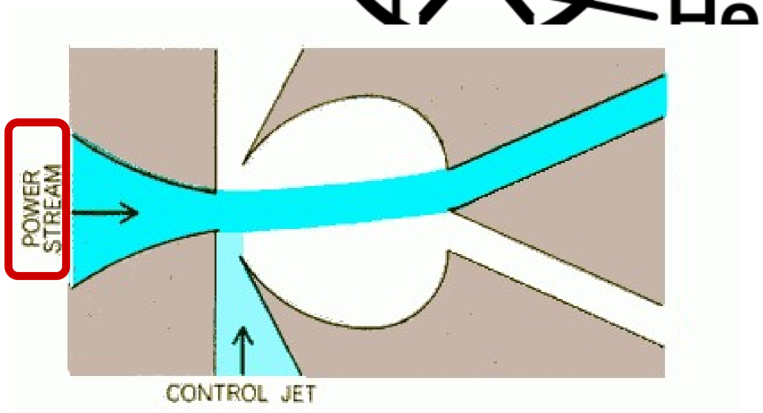




Capstan
(≠winge)

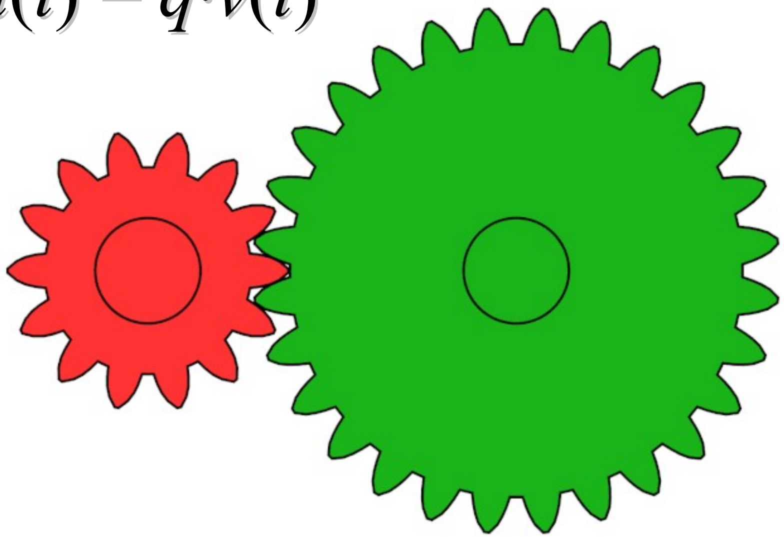




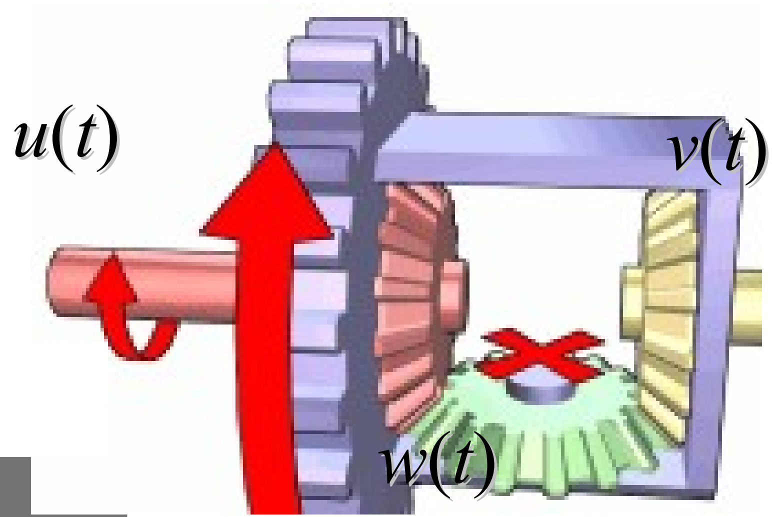


§3 Math of Analog Computing

$u(t) = q \cdot v(t)$

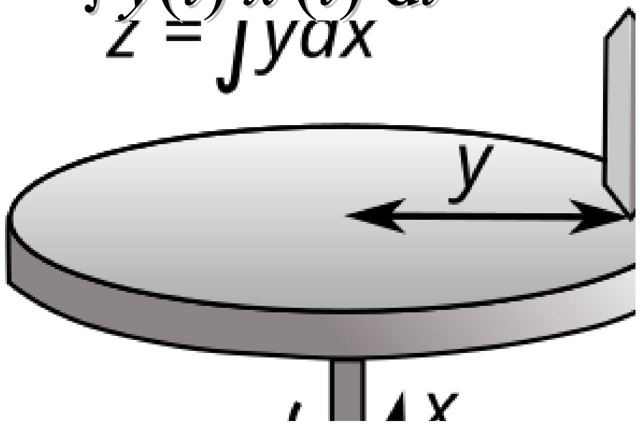


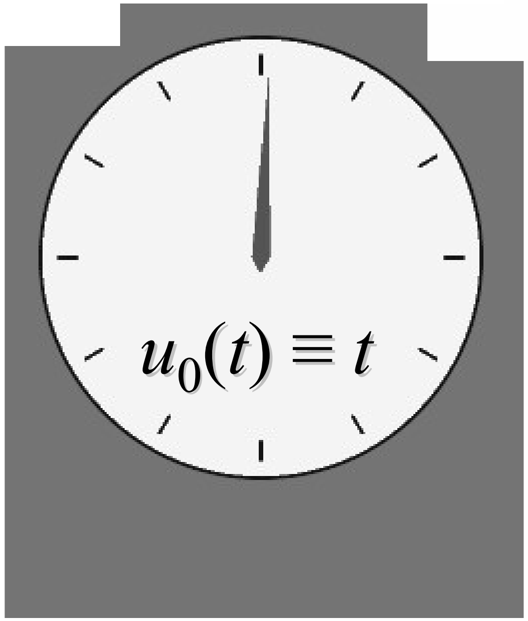
$w(t) = \big(u(t) + v(t)\big)/2$

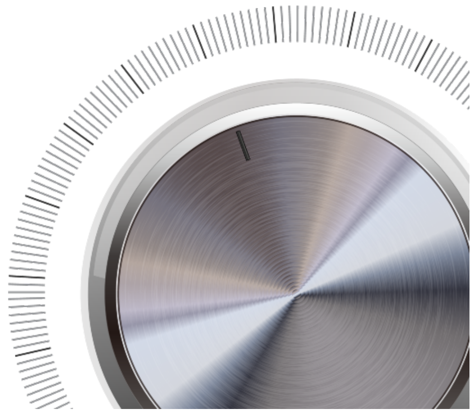


$z(t) = \int y(t)\, \mathrm{d}x(t)$

$= \int y(t) \cdot x'(t)\, \mathrm{d}t$







$c(t) \equiv c$

§3 Shannon: *GPAC = DAEs*

Definition: Let ***GPAC*** denote the least class of functions $u=u(t)$ containing **a)** and **a')** and closed under **b**, **b'**, **c)**.

a) $u_0(t) \equiv t$

a') $c(t) \equiv c$, $c \in \mathbb{R}$

b) $v(t) \to u(t) = q \cdot v(t)$, $q \in \mathbb{Q}$

b') $u(t), v(t) \to w(t) = (u(t) + v(t))/2$

c) $x(t), y(t) \to z(t) = \int y(t) \cdot x'(t)\, \mathrm{d}t$

Theorem: a) If $u_1(t),\ldots,u_n(t)$ are generated by [some configuration of] the Differential Analyzer, then it holds

(*) $u_k'(t) = \sum_{i,j=0..n} c_{ijk} \cdot u_i(t) \cdot u_j'(t)$ for some $c_{ijk} \in \mathbb{R}$

b) If $u_1(t),\ldots,u_n(t)$ satisfy (*) for some $c_{ijk} \in \mathbb{Q}$, then they can be generated by the Differential Analyzer.